Chromatic unstable homotopy, plethories, and the Dieudonné correspondence
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Adams-type spectral sequences for computing $[X, Y]$

The classical, homological Adams spectral sequence:

$$\operatorname{Ext}^{**}_{A_*}(H_*X, H_* Y) \Longrightarrow [X, Y]_*$$

$X, Y$ spectra

$A_*$ dual Steenrod algebra, a commutative $\mathbb{F}_p$-Hopf algebra

$H_*X, H_* Y$ coefficients in $\mathbb{F}_p$, comodules over $A_*$

$\operatorname{Ext}_{A_*}$ of comodules over a coalgebra

$[X, Y]_*$ graded homotopy classes, more realistically $[X, Y^p]$

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Adams-type spectral sequences for computing \([X, Y]\)

The stable Adams-Novikov spectral sequence based on a ring spectrum \(E\):

\[
\text{Ext}_{E_* E}^{**}(E_* X, E_* Y) \Rightarrow [X, Y]_*
\]

\(X, Y\) spectra

\(E_* E\) stable cooperations, a Hopf algebroid

\(E_* X, E_* Y\) comodules over \(E_*\)

\(\text{Ext}_{E_* E}\) of comodules over a Hopf algebroid

\([X, Y]_*\) graded homotopy classes, more realistically \([X, Y_E]\)
Hopf algebroids vs. bialgebras

\((E_*, E_*E)\) is a Hopf algebroid if \(E_*E\) is a flat \(E_*\)-module. A Hopf algebroid is a cogroupoid object in \(E_*\)-algebras.

Alternatively: \(E_*E\) is an \(E_*\)-bimodule with a multiplication

\[
E_*E \otimes_{E_*} E_*E \to E_*E \quad \text{(tensor over } E_* \otimes E_*)
\]

and a comultiplication

\[
E_*E \to E_*E \otimes_{E_*} E_*E \quad \text{(left-right tensor product)}.
\]

It is a bialgebra with respect to two different tensor structures, only one of which is symmetric monoidal. This is the point of view that generalizes to the unstable setting.
Adams-type spectral sequences for computing $[X, Y]$

The unstable spectral sequence based on a ring spectrum $E$:

$$\Ext_{K}^{**}(E_* X, E_* Y) \Longrightarrow [X, Y]_*$$

$X, Y$ CW complexes

$K$ “unstable cooperations”, a “bialgebra” in coalgebras

$E_* X, E_* Y$ comodules over $K$ (in coalgebras), free $E_*$-modules

$\Ext_{K}$ some nonlinear derived functor

$[X, Y]_*$ graded homotopy classes, more realistically $[X, Y_E]$. Fringed, i. e. $[X, Y]_{0,1}$ are only sets/groups.
Adams-type spectral sequences for computing $[X, Y]$

There are adjoint functors

$$\Omega^\infty : E\text{-module spectra} \rightleftarrows \text{Top} : E \wedge \Sigma^\infty(-)_+$$

The associated monad $Y \mapsto E(Y) = \Omega^\infty(E \wedge Y_+)$ gives a bar construction

$$(Y \mapsto E(Y) \Rightarrow E(E(Y)) \cdots)$$

with $\text{Tot} E^\bullet(Y) = Y_E$.

The Bousfield spectral sequence from applying $[X, -]_*$ to the associated tower gives the unstable Adams spectral sequence we are looking for (Bendersky-Curtis-Miller 1978).

It is not obvious how to algebraically describe the $E_2$-term.
Adams-type spectral sequences for computing $[X, Y]$

A comonadic point of view

**Simplifying assumption:** $E_*$ is a graded field, e.g. $E = K(n)$.

Let $\mathcal{K}$ be the comonad $E_* \circ EM$ on $\text{Mod}_{E_*}$. For any space $X$, $E_*X$ is a $\mathcal{K}$-comodule.

**Theorem (BCM 78 for certain connective $E$, Bendersky-Thompson 00 for nonconnective $E$)**

\[ E_2^{s,t} = \text{Ext}_{\mathcal{K}\text{-comodules}}(E_*X, E_*Y) \]
Example

For $E = H\mathbf{F}_p$, $\mathcal{K}$ is the free unstable algebra functor on a graded vector space (rather, its linear dual).

This defines the spectral sequence and identifies its $E_2$-term. But

- It forces us to use the cobar construction – unsuitable for daily use!
- It does not exhibit the bialgebraic structure we saw in the stable case

Aim: Give an algebraic description of the object $\mathcal{P}$ for which $\mathcal{K}$ is the cofree construction.
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**Boardman-Johnson-Wilson 95**: $\mathcal{P}$ is $E_*(E_*)$. This has the following structure:

- An $E_*$-coalgebra because $E_n$ are spaces
- A Hopf algebra because $E_n$ are infinite loop spaces, furthermore a morphism $E_* \rightarrow \text{Hom}_{\text{Hopf}}(\mathcal{P}, \mathcal{P})$
- A *Hopf ring* because $E_*$ is a ring space
- An "enriched" *Hopf ring* by including the action of $E^*(E_*)$ in an ad-hoc way.

**Problem**: what is the (co)composition actually defined on?
A similar algebraic structure has appeared in algebra: *Tall-Wraith monoids* ([Tall-Wraith 70](#)), a.k.a. *plethories*, studied extensively by Borger-Wieland 2005.

Problem: this works for cohomology $E^*(E_*)$, and a profinite topology has to be taken into account.

**Stacey-Whitehouse 09**: give a description of a “completed” version of plethories when $E_*$ is a field (and some more general situations).

B 14: “formal plethories”: definition that works when $E_*$ is a Prüfer domain, informed by algebraic geometry.

But these structures are complex and unwieldy.
Formal groups
(but not what you think)

From now on, assume that $k = E_*$ is a perfect graded field, i.e.:

- $k_0$ is perfect
- If $k$ has period $l$ and of characteristic $p$ then $(l, p) = 1$.

**Definition**

A formal scheme is a directed colimit of functors represented by finite-dimensional $k$-algebras.

\[
\begin{align*}
\text{Top} & \rightarrow \text{formal schemes } / k \\
X & \mapsto \text{Spf } E^*(X) = \colim_{F \subseteq X \text{ finite}} \text{Spec } E^*(F) \\
X \text{ double loop space} & \mapsto \text{formal (abelian) group } / k
\end{align*}
\]
Formal groups

For simplicity, assume even grading, so that everything is commutative.

**Lemma (Fontaine)**

Every formal group $G$ over perfect $k$ splits naturally

$$G = G^c \times G^{ét}, \quad \text{where}$$

- $G^c$ is connected, i.e. $G^c(k') = 0$ for all extensions $k'$ of $k$
- $G^{ét}$ is étale.

There is an equivalence

- étale group schemes $\leftrightarrow$ discrete $\text{Gal}(k)$-modules
  - $G \mapsto \text{colim}_{k < k'} G(k')$
- $\text{Spf map}_{\text{Gal}(k)}(M, \overline{k}) \leftrightarrow M$
Formal groups

If $A$ is an abelian group, considered as a trivial $\text{Gal}(k)$-module, then

$$\text{Spf map}_{\text{Gal}(k)}(A, \bar{k}) = \text{Spf map}(A, k) = A$$

is the constant group scheme.

**Observation:** If $X$ is a space, $E^*(X)/\text{nil} \cong E^*(\pi_0 X)$ has trivial $\text{Gal}(k)$-action. In particular, when $X$ is a commutative, associative grouplike $H$-space, $(\text{Spf } E^*(X))^{\text{ét}}$ is always constant.

**Definition**

A *cohomological formal group* is a formal group with constant étale part.
The category of cohomological formal groups

Theorem (B)

The category of cohomological formal groups

- is abelian
- has all colimits and limits, and directed colimits are exact
- has a generator and a cogenerator
- is well-powered and co-well-powered.

Corollary (Freyd)

Any functor from cohomological formal groups to another category $\mathcal{C}$ that preserves limits, has a left adjoint. Any functor that preserves colimits has a right adjoint.

In particular, the forgetful functor to formal schemes has a left adjoint $\text{Fr}$. 
Tensor products of formal groups

Corollary (Goerss 1999)

The category of cohomological formal groups is closed monoidal with respect to a tensor product $\otimes$ classifying bilinear maps of formal groups.

Example

$A \otimes B \cong A \otimes B$

Example

$\text{Fr}(X \times Y) \cong \text{Fr}(X) \otimes \text{Fr}(Y)$

In characteristic 0, every connected formal group is the zero component of a free formal group (Milnor-Moore), so this is nearly a complete description.
Definition

Let \( l \) be a graded commutative ring. A formal \( l \)-algebra scheme is a formal group \( A \) with \( l \to A, A \otimes A \to A \) making it into a functor from \( k \)-algebras to \( l \)-algebras.

Example

If \( F \) is a commutative ring spectrum, \( \text{Spf} \, E^*F_* \) is a cohomological \( F_* \)-algebra scheme.

Lemma

Composition gives:

\[
\{ \text{f. } l\text{-alg schemes}/k \} \times \{ \text{f. schemes}/k \} \to \{ \text{f. schemes}/l \}
\]

\[
\{ \text{f. } l\text{-alg schemes}/k \} \times \{ \text{f. } k\text{-alg schemes}/k \} \to \{ \text{f. } l\text{-alg schemes}/k \}
\]
Formal plethories

Lemma

Composition gives:

\[ \{ \text{f. l-alg schemes/}k \} \times \{ \text{f. schemes/}k \} \rightarrow \{ \text{f. schemes/}l \} \]

\[ \{ \text{f. l-alg schemes/}k \} \times \{ \text{f. k-alg schemes/}k \} \rightarrow \{ \text{f. l-alg schemes/}k \} \]

In particular, composition \( \circ \) is a non-symmetric monoidal structure on \( k \)-algebra schemes over \( k \), and schemes over \( k \) are tensored over it.

Definition

A \textbf{formal plethory} is a cohomological formal \( k \)-algebra scheme \( P \) with a comonoid structure

\[ P \rightarrow P \circ P, \ P \rightarrow \text{id} \]

A (left) comodule over a formal plethory \( P \) is a formal scheme \( X \) with a coaction \( X \rightarrow P \circ X \).

Thus: algebra for \( \otimes \), coalgebra for \( \circ \).
Formal plethories

**Example**

$\text{Spf } E^*_E$ is a formal plethory, and $\text{Spf } E^*(X)$ is a comodule over it for any space $X$.

**Theorem (2014, “nonlinear Künneth theorem”)**

For $E, F$ commutative ring spectra, $E_*$ a graded field, and $X$ a space, there is an isomorphism

$$\text{Spf}(E^*(F(X))) = \text{Spf } E^*(\Omega^\infty(F \wedge X)_+) \cong \text{Spf } E^*_E \circ \text{Spf } F^*X$$

**Corollary**

The functor $C \mapsto \text{Spf } C^*$ from coalgebras to formal schemes extends to an equivalence between $\mathcal{K}$-coalgebras and $\text{Spf } E^*_E$-comodules. Thus

$$E_2 = \text{Ext}_{\text{Spf } E^*_E \text{-comod}}(\text{Spf } E^*(S^*), \text{Spf } E^*(X)) \Rightarrow \pi_*X^E.$$
Corollary

The functor \( C \mapsto \text{Spf} \, C^* \) from coalgebras to formal schemes extends to an equivalence between \( K \)-coalgebras and \( \text{Spf} \, E^*E^*_\text{-comodules} \). Thus

\[
E_2 = \text{Ext}_{\text{Spf} \, E^*E^*_\text{-comod}}(\text{Spf} \, E^*(S^*), \text{Spf} \, E^*(X)) \Rightarrow \pi_*X^\wedge_E.
\]

+ Fully algebraic description of \( E_2 \)
- Cobar construction still the only obvious resolution – what are injective \( \text{Spf} \, E^*E^*_\text{-comodules} \)?
From now on, \( \text{char}(k) = p > 0 \). Semi-classically:

**Theorem**

Let \( l \) be a \( \mathbb{Z}_p \)-algebra. Then there is an equivalence of abelian categories

\[
\left\{ \text{discrete cohomological } \mathcal{O}_l \text{-module schemes over } k \right\} \xrightarrow{D} \left\{ \text{Modules } M \text{ over } \mathcal{R} = \mathcal{W}(k) \otimes \mathbb{Z}_p \langle F, V \rangle / (FV - p) \text{ s.t. } M = M_0 \oplus M_c \text{ where } V|_{M_0} = \text{id}, V|_{M_c} \text{ nilpotent} \right\}
\]

**Remark:** If one drops the cohomologicality requirement, \( V \) has finite order on \( M_0 \) instead of order 1.
The Dieudonné correspondence

Example

\[ D(A) = A \quad \text{with} \quad V = \text{id}, \quad F = p \]

Example

\[ D(\hat{\mathbb{G}}_a) = k[V^{\pm 1}]/k[V] = \langle \bullet \leftrightarrow \bullet \leftrightarrow \cdots \rangle, \quad F = 0 \]

Example (and Proposition)

For the free formal group \( \text{Fr}(X) \) on a scheme \( X = \text{Spec} \, A \), \( A \) finite:

\[ D(\text{Fr}(X)) = \text{Hom}(\text{CW}(A), \text{CW}(k)) \]

\( \text{CW}(A) \) = \( p \)-typical “co-Witt vectors” are to \( \mathcal{W}(A) \) what \( \mathbb{Z}/p^\infty \) is to \( \mathbb{Z}_p \).
Theorem (Goerss ’99, Buchstaber-Lazarev ’07)

Given two formal $l$-modules $M, N$,

$$D(M \otimes N) \cong D(M) \boxtimes D(N),$$

where

$$A \boxtimes B = \mathcal{R} \otimes W(k)[V] (A \otimes B)/ \sim,$$

$$Fx \otimes Va \otimes b \sim x \otimes a \otimes Fb, \quad Fx \otimes a \otimes Vb \sim x \otimes Fa \otimes b$$

In particular, formal $l$-algebra schemes correspond to $\boxtimes$-algebras $A$ with unit $l \to A$. 
The evaluation product

**Theorem (B)**

**The evaluation product**

\[
\{\text{cohomological formal } l\text{-modules}\} \times \{k\text{-algebras}\} \xrightarrow{\text{ev}} \{l\text{-modules}\}
\]

satisfies \(G(A) = D(G) \circ A\), where

\[
M \circ A = \text{Tor}^{W(k)}(M, CW(A))^{F,V} = \ker \left( \begin{array}{c}
\text{Tor}(F, \text{id}) - \text{Tor}(\text{id}, V) \\
\text{Tor}(V, \text{id}) - \text{Tor}(\text{id}, F)
\end{array} \right)
\]

**Note:** \(\circ\) is linear on the left, but not on the right.
The evaluation product, simplified

If $l$ is an $\mathbb{F}_p$-algebra, there is a simpler description.

**Theorem**

If $l$ is an $\mathbb{F}_p$-algebra, $M \circ A \cong M \otimes_{\mathbb{F}} W(A)$, where

$$M \otimes_{\mathbb{F}} W(A) = M \otimes W(A) / (Fm \otimes a - m \otimes Va)$$

and

$$M \otimes_{\mathbb{F}}^{\mathcal{V}} W(A) = \ker(V \otimes \text{id} - \text{id} \otimes F) \text{ on } M \otimes_{\mathbb{F}} W(A)$$
The evaluation product, simplified

\[ M \otimes_F W(A) = M \otimes W(A) / (Fm \otimes a - m \otimes Va) \]

in more explicit terms:
Define polynomials \( c_i(x, y) \) inductively by

\[
x^{p^n} + y^{p^n} = c_0(x, y)^{p^n} + pc_1(x, y)^{p^{n-1}} + \cdots + p^n c_n(x, y)
\]

\[
c_0(x, y) = x + y, \quad c_1(x, y) = \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^i y^{p-i}, \ldots
\]

Then \( M \otimes_F W(A) \) is generated by symbols \((m, a)\) modulo left linearity and

\[
(m, a) + (m, b) \sim \sum_{i=0}^{\infty} (F^i m, c_i(a, b))
\]
The composition product

**Theorem (B)**

Given a two $k$-algebra scheme $F$ and $G$, $D(G \circ F) = D(F) \circ D(G)$. Here,

$$M \circ N = (M \otimes_F^V N),$$

where $M \otimes_F N$ is generated by $(m, n)$ modulo left linearity and

$$(m, n) + (m, n') = (m, n + n') + (Fm, Vc_1(n, n')) + \cdots,$$

$$(M \otimes_F^V N) = \ker(V \otimes \text{id} - \text{id} \otimes V(-)^p).$$

The $R$-module structure on $M \circ N$ is given by $V(m, n) = (m, Vn)$ and $F(m, n) = (m, Fn) + (Fm, n^p)$. The multiplication is componentwise.
The plethory for $K(1)$, $p > 2$

Classical stable computation:

$$K(n)\big(K(n)\big) = P(b_1, b_2, \ldots)/(b_i^p - v_n^2 b_i) \otimes \bigwedge (a_0, \ldots, a_{n-1}).$$

Make this an $R$-module by defining $F = 0$, $V(a_i) = a_{i-1}$, $V(b_i) = b_{i-1}$. Then it becomes a $(\boxtimes, \circ)$-bialgebra.

Let $P = D(Spf K(n)^* K(n)^*)$ be the plethory for $K(n)$ under the Dieudonné correspondence.

Stabilization: $P \to K(n)\big(K(n)\big)$ is surjective [Kuhn, Wilson].

**Theorem**

*There is a short exact sequence of $(\boxtimes, \circ)$-bialgebras*

$$k \to k[e]/(e^{2p-1} - e) \to P \to K(1)\big(K(1)\big) \to k.$$
Theorem

There is a short exact sequence of \((\boxtimes, \circ)\)-bialgebras

\[ k \to k[e]/(e^{2p-1} - e) \to P \to K(1) \ast K(1) \to k. \]

More precisely, \(P \cong k[e]/(e^{2p-1} - e) \otimes K(1) \ast K(1)\) as algebras,

- \(|e| = (1, 1), \ |a_0| = (2, 1), \ |b_i| = (2p^i, 2)\)
- \(V(b_i) = b_{i-1}, \ V(b_1) = e^2 =: b_0, \ V(e) = V(a_0) = 0\)
- \(F(a_0) = (1 - v_1^{-1} e^{2p-2} v_1) a_0, \ F(b_i) = 0 = F(e)\)
- \(\psi(e) = e \circ e, \ \psi(a_0) = a_0 \circ e_1 + e_1^2 \circ a_0, \ \sum_{n \geq 0} F_\psi(b_n) = \sum_{i,j \geq 0} F b_i^j \circ b_j\)