The $K(2)$–Local Picard Group at $p = 2$

(Joint with Bobkova, Goerss and Henn)

Agnès Beaudry

University of Colorado

August 18, 2016

Picard Group of a Category

Definition

For $(\mathcal{C}, \otimes, I)$ a symmetric monoidal category,

$$\mathrm{Pic}(\mathcal{C}) = \frac{\{\text{invertible objects}\}}{\text{isomorphisms}}$$

Remark

An element of $X \in \mathrm{Pic}(\mathcal{C})$ gives an automorphism of $\mathcal{C}$:

$$A \in \mathcal{C} \mapsto A \otimes X$$

Example ($\mathcal{C} = \mathcal{O}_X$-modules)

Let X be a scheme.

$$\begin{aligned} \mathrm{Pic}(\mathcal{C}) &= \frac{\{\text{locally free } \mathcal{O}_X\text{–modules of rank one }\}}{\text{isomorphisms}} \\ &\cong H^1(X, \mathcal{O}_X^*) \end{aligned}$$

Example ($\mathcal{C} = G$-R-modules)

Let R be a local Noetherian ring and $G \circlearrowright R$.

$$\begin{aligned} \mathrm{Pic}(\mathcal{C}) &= \frac{\{G\text{-}R\text{-modules that are free rank one over } R\}}{\text{isomorphisms}} \\ &\cong H^1(G, R^*) \end{aligned}$$

Example ($\mathcal{C} = \mathrm{hSp}_E$: Homotopy category of E-local spectra)

If E is S^0, $H\mathbb{Q}$, $H\mathbb{F}_p$, $H\mathbb{Z}_{(p)}$, or $K_{(p)}, K_p$ with p odd, then

$$\mathrm{Pic}(\mathrm{hSp}_E) \cong \mathbb{Z}\left\langle S^1_E \right\rangle.$$

For $p = 2$

$$\mathrm{Pic}(\mathrm{hSp}_{K_2}) \cong \mathbb{Z}\left\langle S^1_{K_2} \right\rangle \oplus \mathbb{Z}/2\left\langle DQ_{K_2} \right\rangle.$$

DQ :

Further,

$$\mathrm{Pic}(\mathrm{hSp}_{K\mathbb{F}_p}) \cong \begin{cases} \mathbb{Z}_p \times \mathbb{Z}/2(p-1) \\ \mathbb{Z}_2 \times \mathbb{Z}/2 \times \mathbb{Z}/4. \end{cases}$$

Notation

$$K(1) = K\mathbb{F}_p \qquad E_1 = K_p \qquad \mathrm{Pic}_1 = \mathrm{Pic}(\mathrm{hSp}_{K(1)})$$

Theorem (Hopkins-Mahowald-Sadofsky)

The following are equivalent

1. $X \in \mathrm{Pic}_1$
2. $K(1)_* X$ *is of rank one as a* $K(1)_*$*–module.*
3. $(E_1)_* X = \pi_*(E_1 \wedge X)_{K(1)}$ *is free of rank one as* $(E_1)_*$*–module.*

Remark

For $\mathbb{G}_1 = \mathbb{Z}_p^\times$ the Adams operations, $(E_1)_* X$ is a $\mathbb{G}_1$-$(E_1)_*$-module.

$$\mathrm{Pic}_1^{alg} = \mathrm{Pic}(\mathbb{G}_1\text{-}(E_1)_*\text{-modules})$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \kappa_1 & \longrightarrow & \mathrm{Pic}_1 & \longrightarrow & \mathrm{Pic}_1^{alg} & \longrightarrow & 0 \\ & & & & X & \longmapsto & (E_1)_* X & & \end{array}$$

For $M \in \mathrm{Pic}_1^{alg}$, $M \cong (E_1)_{*+\epsilon}$ for $\epsilon = 0, 1$. There is a non-split extension

$$\begin{tikzcd} 0 \arrow[r] & (\mathrm{Pic}_1^{alg})_0 \arrow[r] & \mathrm{Pic}_1^{alg} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \\ & & M \arrow[r, mapsto] & \epsilon & \end{tikzcd}$$

and

$$(\mathrm{Pic}_1^{alg})_0 \cong H^1(\mathbb{G}_1, (\pi_0 E_1)^*) \cong H^1(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$$

So

$$\mathrm{Pic}_1^{alg} \cong \begin{cases} \mathbb{Z}_p \times \mathbb{Z}/2(p-1) & p \text{ odd} \\ \mathbb{Z}_2 \times (\mathbb{Z}/2)^2 & p = 2. \end{cases}$$

Theorem (Hopkins, Mahowald, Sadofsky)

If p is odd, $\mathrm{Pic}_1 \cong \mathrm{Pic}_1^{alg}$.

$$\begin{tikzcd} 0 \arrow[r] & \kappa_1 \arrow[r] \arrow[d, equal] & \mathrm{Pic}_1 \arrow[r, "\cong"] & \mathbb{Z}_p \times \mathbb{Z}/2(p-1) \arrow[r] & 0 \\ & 0 & & & \end{tikzcd}$$

For $M \in \mathrm{Pic}_1^{alg}$, $M \cong (E_1)_{*+\epsilon}$ for $\epsilon = 0, 1$. There is a non-split extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathrm{Pic}_1^{alg})_0 & \longrightarrow & \mathrm{Pic}_1^{alg} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\ & & & & M & \longmapsto & \epsilon & & \end{array}$$

and

$$(\mathrm{Pic}_1^{alg})_0 \cong H^1(\mathbb{G}_1, (\pi_0 E_1)^*) \cong H^1(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$$

So

$$\mathrm{Pic}_1^{alg} \cong \begin{cases} \mathbb{Z}_p \times \mathbb{Z}/2(p-1) & p \text{ odd} \\ \mathbb{Z}_2 \times (\mathbb{Z}/2)^2 & p = 2. \end{cases}$$

Theorem (Hopkins, Mahowald, Sadofsky)

If $p = 2$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \kappa_1 & \longrightarrow & \mathrm{Pic}_1 & \longrightarrow & \mathrm{Pic}_1^{alg} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z}/2\left\langle DQ_{K(1)}\right\rangle & & \mathbb{Z}_2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 & & \mathbb{Z}_2 \times (\mathbb{Z}/2)^2 & & \end{array}$$

Theorem (Hopkins, Mahowald, Sadofsky)

If $p = 2$

$$\begin{tikzcd} 0 \arrow[r] & \kappa_1 \arrow[r] \arrow[d, equal] & \mathrm{Pic}_1 \arrow[r] \arrow[d, equal] & \mathrm{Pic}_1^{alg} \arrow[r] \arrow[d, equal] & 0 \\ & \mathbb{Z}/2\left\langle DQ_{K(1)}\right\rangle & \mathbb{Z}_2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 & \mathbb{Z}_2 \times (\mathbb{Z}/2)^2 & \end{tikzcd}$$

Remarks on κ_1

- $(E_1 \wedge DQ)_{K(1)} \simeq E_1$
- $C_2 = (\pm 1) \subset \mathbb{Z}_2^\times = \mathbb{G}_1$
- $E_1^{hC_2} = K_2^{hC_2} \simeq KO_2$
- $(E_1^{hC_2} \wedge DQ)_{K(1)} \simeq \Sigma^4 E_1^{hC_2} \not\simeq E_1^{hC_2}$
- $E_1^{hC_2}$ or C_2 *detects* $DQ_{K(1)}$.

Higher K–Theories

Fix a prime p.

- $K(n) =$ Morava K-theory

$$K(n)_* \cong \mathbb{F}_{p^n}[u^{\pm 1}]$$

  For example, $K(1) = K\mathbb{F}_p$.

- $E_n =$ Morava E–theory

$$(E_n)_* \cong \mathbb{Z}_{p^n}[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}]$$

  For example, $E_1 = K_p$.

- $\mathbb{G}_n =$ Morava Stablizer group (higher Adams Operations)

Notation

$K(n)$ = Higher $K\mathbb{F}_p$ $\quad E_n$ = Higher K_p $\quad \mathbb{G}_n$ = Higher Adams Ops

Definition

$\mathrm{Pic}_n = \mathrm{Pic}(\mathrm{hSp}_{K(n)})$

Why care?

- ◯ hSp is built from the $\mathrm{hSp}_{K(n)}$ (for all p and n).
- □ Pic_n give automorphisms of $\mathrm{hSp}_{K(n)}$
- $\Diamond$ $I_n \in \mathrm{Pic}_n$ where I_n is the Brown–Comenetz dual of $S_{K(n)}$

Notation

$K(n) =$ Higher $K\mathbb{F}_p$ $\quad E_n =$ Higher K_p $\quad \mathbb{G}_n =$ Higher Adams Ops

Theorem (Hopkins-Mahowald-Sadofsky)

The following are equivalent

1. $X \in \mathrm{Pic}_n$
2. $K(n)_*X$ *is of rank one as a* $K(n)_*$*–module.*
3. $(E_n)_*X = \pi_*(E_n \wedge X)_{K(n)}$ *is free of rank one as* $(E_n)_*$*–module.*

Remark

For $\mathbb{G}_n$ the higher Adams operations, $(E_n)_*X$ is a $\mathbb{G}_n$-$(E_n)_*$-module.

$$\mathrm{Pic}_n^{alg} = \mathrm{Pic}(\mathbb{G}_n\text{-}(E_n)_*\text{-modules})$$

$$0 \longrightarrow \kappa_n \longrightarrow \mathrm{Pic}_n \longrightarrow \mathrm{Pic}_n^{alg} \longrightarrow ?$$

Overview

n	p	Pic_n	$Pic_{n,alg}$	κ_n	Who
1	$\geqslant 3$	$\mathbb{Z}_p \times \mathbb{Z}/2(p-1)$	$\mathbb{Z}_p \times \mathbb{Z}/2(p-1)$	0	HoMSa
1	2	$\mathbb{Z}_2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$	$\mathbb{Z}_2 \times (\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	HoMSa
2	$\geqslant 5$	$\mathbb{Z}_p^2 \times \mathbb{Z}/2(p-1)$	$\mathbb{Z}_p^2 \times \mathbb{Z}/2(p-1)$	0	Ho
2	3	$\mathbb{Z}_3^2 \times \mathbb{Z}/16 \times (\mathbb{Z}/3)^2$	$\mathbb{Z}_3^2 \times \mathbb{Z}/16$	$(\mathbb{Z}/3)^2$	Kr, GHnMR, KmSh
2	2	?	$\mathbb{Z}_2^2 \times \mathbb{Z}/3 \times (\mathbb{Z}/2)^3$	$\bigstar$	Hn, BeBoGHn
$\geqslant 3$	$\gg n$	?	?	0	HoMSa
$\geqslant 3$	$\geqslant 3$	?	?	p-gp	Ha

Be = Beaudry Bo = Bobkova G= Goerss Ha = Heard
Hn = Henn Ho = Hopkins M = Mahowald R = Rezk
Sa = Sadofsky Sh = Shimomura Km = Kamiya Kr = Karamanov

Remark

At $n = p = 2$, we do not yet know if $\mathrm{Pic}_2 \to \mathrm{Pic}_2^{alg}$ is surjective.

Theorem (★BBGH)

The group κ_2 *has a filtration*

$$\kappa_2^{\mathbb{G}_2^1} \subset \kappa_2^F \subset \kappa_2$$

such that

$$\kappa_2^{\mathbb{G}_2^1} \cong \mathbb{Z}/4 \oplus \mathbb{Z}/8 \qquad \kappa_2^F/\kappa_2^{\mathbb{G}_2^1} \cong \mathbb{Z}/2 \qquad \kappa_2/\kappa_2^F \cong \mathbb{Z}/8.$$

In particular, $|\kappa_2| = 2^9$.

Filtration by subgroups $G \subseteq \mathbb{G}_n$

- $$\begin{tikzcd} 0 \arrow[r] & \kappa(G) \arrow[r] & \mathrm{Pic}(E_n^{hG}\text{-mod}) \arrow[r] \arrow[d, phantom, "||"] & \mathrm{Pic}(G\text{-}(E_n)_*\text{- mod}) \arrow[d, phantom, "||"] \\ & & \mathrm{Pic}(G) & \mathrm{Pic}^{alg}(G) \end{tikzcd}$$
- $X \mapsto E_n^{hG} \wedge X$ gives a map:

$$\kappa_n \xrightarrow{i_G} \kappa(G)$$

with kernel $\kappa_n^G = \{X \mid E_n^{hG} \wedge X \simeq E_n^{hG}\}$.
- If $G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n \subseteq \mathbb{G}_n$, then $\kappa_n^{G_n} \subseteq \kappa_n^{G_{n-1}} \subseteq \ldots \subseteq \kappa_n^{G_0} \subseteq \kappa_n$.

Subgroups of $\mathbb{G}_n$

There is a group homomorphism $\det : \mathbb{G}_n \to \mathbb{Z}_p^\times$.

- $\mathbb{G}_n^1 = \ker(\mathbb{G}_n \to \mathbb{Z}_p^\times/\mu_p)$ where $\mathbb{Z}_p^\times/\mu_p \cong \mathbb{Z}_p$.
- $\mathbb{G}_n \cong \mathbb{G}_n^1 \rtimes \mathbb{Z}_p$
- F a maximal finite subgroup (containing the p–torsion).

Filtration by subgroups $G \subseteq \mathbb{G}_n$

- $$\begin{tikzcd} 0 \arrow[r] & \kappa(G) \arrow[r] & \mathrm{Pic}(E_n^{hG}\text{-mod}) \arrow[r] \arrow[d, phantom, "||"] & \mathrm{Pic}(F\text{-}(E_n)_*\text{- mod}) \arrow[d, phantom, "||"] \\ & & \mathrm{Pic}(G) & \mathrm{Pic}^{alg}(G) \end{tikzcd}$$

  item $X \mapsto E_n^{hG} \wedge X$ gives a map:

  $$\kappa_n \xrightarrow{i_G} \kappa(G)$$

  with kernel $\kappa_n^G = \{X \mid E_n^{hG} \wedge X \simeq E_n^{hG}\}$.
- If $G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n \subseteq \mathbb{G}_n$, then $\kappa_n^{G_n} \subseteq \kappa_n^{G_{n-1}} \subseteq \ldots \subseteq \kappa_n^{G_0} \subseteq \kappa_n$.

Example $(n = 1)$

For $F = C_2$ in $\mathbb{G}_1 \cong \mathbb{Z}_2^\times$, $\kappa_1 \cong \kappa(C_2) \cong \mathbb{Z}/2$ and $\kappa_1^{C_2} = 0$

$$E_1^{hC_2} \wedge DQ_{K(1)} \simeq \Sigma^4 E_1^{hC_2} \not\simeq E_1^{hC_2}$$

since $E_1^{hC_2} \simeq KO_2$ is 8–periodic, but not 4–periodic.

Filtration by subgroups $G \subseteq \mathbb{G}_n$

- $$\begin{tikzcd} 0 \arrow[r] & \kappa(G) \arrow[r] & \mathrm{Pic}(E_n^{hG}\text{-mod}) \arrow[r] \arrow[d, phantom, "||"] & \mathrm{Pic}(F\text{-}(E_n)_*\text{- mod}) \arrow[d, phantom, "||"] \\ & & \mathrm{Pic}(G) & \mathrm{Pic}^{alg}(G) \end{tikzcd}$$
- $X \mapsto E_n^{hG} \wedge X$ gives a map:

$$\kappa_n \xrightarrow{i_G} \kappa(G)$$

with kernel $\kappa_n^G = \{X \mid E_n^{hG} \wedge X \simeq E_n^{hG}\}$.
- If $G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n \subseteq \mathbb{G}_n$, then $\kappa_n^{G_n} \subseteq \kappa_n^{G_{n-1}} \subseteq \ldots \subseteq \kappa_n^{G_0} \subseteq \kappa_n$.

Theorem (Kr, GHMR, KmSh)

Let $p = 3$ and $F \subset \mathbb{G}_2$ be a maximal finite subgroup containing the 3–torsion.

$$\kappa_2 \cong \mathbb{Z}/3 \times \mathbb{Z}/3 \cong \kappa(F) \times \kappa_2^F$$

and $\kappa_2^F = \kappa_2^{\mathbb{G}_2^1}$.

Subgroups of $\mathbb{G}_n$

There is a group homomorphism $\det : \mathbb{G}_n \to \mathbb{Z}_p^\times$.

- $\mathbb{G}_n^1 = \ker(\mathbb{G}_n \to \mathbb{Z}_p^\times/\mu_p)$ where $\mathbb{Z}_p^\times/\mu_p \cong \mathbb{Z}_p \langle \psi \rangle$.
- $\mathbb{G}_n \cong \mathbb{G}_n^1 \rtimes \mathbb{Z}_p$

Remark (GHMR, Westerland)

There is a fiber sequence:

$$S_{K(n)} \longrightarrow E_n^{h\mathbb{G}_n^1} \xrightarrow{\psi - 1} E_n^{h\mathbb{G}_n^1}$$

If $\lambda \in \pi_0 E_n^{h\mathbb{G}_n^1}$ and $(E_n)_* \lambda = 1$. Let X_λ be the fiber

$$X_\lambda \longrightarrow E_n^{h\mathbb{G}_n^1} \xrightarrow{\psi - \alpha} E_n^{h\mathbb{G}_n^1}.$$

Then, $X_\lambda \in \kappa_n^{\mathbb{G}_n^1}$.

Strategy $F \subseteq \mathbb{G}_2^1 \subseteq \mathbb{G}_2$ at $p = 2$

Use $\kappa_2 \longrightarrow \kappa(\mathbb{G}_2^1) \longrightarrow \kappa(F)$

- $\kappa(F) \cong \mathbb{Z}/8$ (Heard-Mathew-Stojanoska)
- Prove that $\kappa_2 \xrightarrow{i_F} \kappa(F)$ is surjective.
  - Use the relationship $I_2 E_2^{hG_{48}} \simeq DE_2^{hG_{48}} \wedge I_2$.
- Prove that $\kappa(\mathbb{G}_2^1)/\ker(\kappa(\mathbb{G}_2^1) \to \kappa(F)) \cong \mathbb{Z}/2$
  - Use obstruction theory for the Bo-GHMR duality resolution

$$(\mathrm{Gal})_+ \wedge E_2^{h\mathbb{G}_2^1} \to E_2^{hF} \to E_2^{hH} \to E_2^{hH} \to \Sigma^{48} E_2^{hF}.$$

- Prove that $\kappa_2^{\mathbb{G}_2^1} \cong \mathbb{Z}/4 \oplus \mathbb{Z}/8$
  - By proving $\pi_0 E_2^{h\mathbb{G}_1} \cong \mathbb{Z}_2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8$.
- Prove that $\kappa_2 \xrightarrow{i_{\mathbb{G}_2^1}} \kappa(\mathbb{G}_2^1)$ is surjective.
  - Descent technology of Heard-Mathew-Stojanoska.

Theorem (BBGH)

$|\kappa_2| = 2^9$ *and*

$$\kappa_2^{\mathbb{G}_2^1} \cong \mathbb{Z}/4 \oplus \mathbb{Z}/8 \qquad \kappa_2^F/\kappa_2^{\mathbb{G}_2^1} \cong \mathbb{Z}/2 \qquad \kappa_2/\kappa_2^F \cong \mathbb{Z}/8.$$

Questions

- What is the group structure of κ_2?
- Recall: at $p = 3$, $\kappa_2 \cong \kappa(F) \times \kappa_2^F$.
  - * Is this true at $p = 2$? I.e., is $\kappa_2 \xrightarrow{i_F} \kappa(F)$ split?
- Recall: $DQ_{K(1)}$ generates $\kappa_1 = \kappa(C_2)$.
  - * Is there a finite X such that $X_{K(2)}$ generates $\kappa(F)$? ($p = 2, 3$)
- Recall: at $n = p-1$, $EO_{p-1} = E_{p-1}^{hF}$ and $\kappa(F) \cong \mathbb{Z}/p$ (HeMaSto).
  - * Is $\kappa_n \to \kappa(F)$ split surjective at $n = p-1$?

Thank you! And now for Tilman Bauer... (P.S. sseq is GREAT!)