Intersection Homotopy Theory

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Use techniques of homotopical algebra to study topological invariants of singular spaces.
These topological invariants coming from intersection cohomology are not homotopy invariants in general.

The case of complex algebraic varieties

Let V be a complex algebraic variety.
Let $Sh^*_R(V)$ be the category of cochain complexes of sheaves of R-modules.
To any $\mathbf{F}^* \in Sh^*_R(V)$ we associate hypercohomology groups :

$$\mathbb{H}^*(V; \mathbf{F}^*).$$

Two important examples

(1) When we consider the constant sheaf $\mathbf{R}$ we get

$$\mathbb{H}^*(V; \mathbf{R}) \cong H^*(V; R)$$

(2) M. Goresky and R. MacPherson introduced **intersection complexes** $\mathbf{IC}^*_{\overline{p}} \in Sh^*_R(V)$ and intersection cohomology :

$$IH^*_{\overline{p}}(V; R) := \mathbb{H}^*(V; \mathbf{IC}^*_{\overline{p}})$$

for any perversity $\overline{p}$ (a sequence of integers).

Why should we care about $\mathbf{IC}^*_{\bar{p}}$?

(1) They restore Poincaré duality for singular spaces. They are essential to construct characteristic numbers and classes for singular spaces (L-classes and Wu classes). Applications to surgery of topological manifolds (D. Sullivan, J. Morgan, M. Goresky, R. MacPherson, P. Siegel).
(2) They are the building blocks of the category of perverse sheaves (Beilinson, Bernstein and Deligne).
(3) Related to L^2-cohomology (J. Cheeger).

From cohomology to homotopy theory

(1) Singular cohomology is natural.
(2) It factors through $Ho(Top)$.
(3) It is representable.
(4) It is a commutative graded algebra.
(5) It is the cohomology of a natural E_∞-dg algebra : $C^*(X;R)$.
(6) One can recover under some "nice" asumptions the rational homotopy type and the p-adic homotopy types from this cochain algebra.

Intersection homotopy theory

Reformulation of questions and problems of M. Goresky, R. MacPherson and G. Friedman, M. Hovey, J. McClure.
(1) Define a category of stratified spaces together with a notion of stratified homotopy such that intersection cohomology is representable in the homotopy category of stratified spaces.
(2) $\{IH^*_{\overline{p}}(X;R)\}_{\overline{p}}$ is a perverse graded algebra. Is it the cohomology of a natural perverse E_∞-dg algebra ? When R is a field of characteristic zero, is it the cohomology of a natural perverse commutative-dg algebra ?
(3) Set the foundations of rational intersection homotopy theory à la Sullivan and study the formality of complex projective varieties.

Stratified spaces

Definition :

Let X be a filtered topological space $X_0 \subset X_1 \subset \cdots \subset X_n$ such that X_i is a closed subset.
The **formal dimension** of X is n.
The connected components of $X_i - X_{i-1}$ are called **the strata** of codimension $n - i$.
The strata of codimension zero are the **regular** strata.
The space X is **stratified** if it satisfies the frontier condition :
For any pair of strata S and S' such that $S \cap \overline{S'} \neq \emptyset$ then $S \subset \overline{S'}$.

Stratified morphisms

Definition :

A continuous map

$$f : (X, \{X_i\}_{0\leq i\leq n}) \to (Y, \{Y_i\}_{0\leq i\leq n})$$

is **stratum preserving** if $f^{-1}(Y_i) = X_i$.
We denote by $\mathbf{Strat_n}$ the category of stratified spaces of formal dimension n and stratum preserving maps.

Topological pseudomanifolds

Definition :

A **topological pseudomanifold** of dimension n is a stratified space X of formal dimension n such that :

(1) Each stratum of codimension i is a topological manifold of dimension $n-i$.

(2) No stratum of codimension 1.

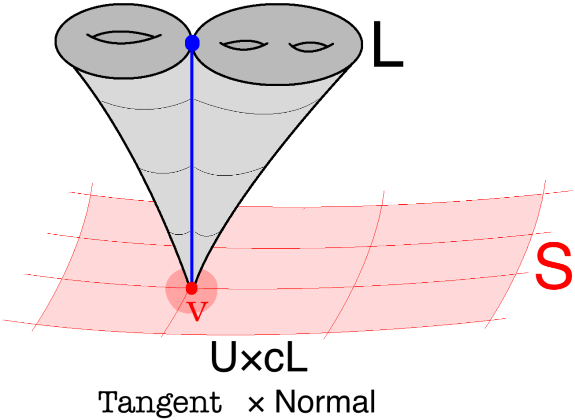
(3) Each point $x \in X_i - X_{i-1}$ has a conical neighbourhood :

$$U_x \cong \mathbb{R}^i \times cL^{n-i-1}$$

where L^{n-i-1} is a topological pseudomanifold of dimension $n-i-1$.

(4) X is oriented if the regular part is oriented.

Local chart



Examples

(1) Complex algebraic varieties (H. Whitney).
(2) Quotients M/G.
(3) Let $M = \partial W$ we can form a space with isolated singularities $X = W \cup_M cM$.

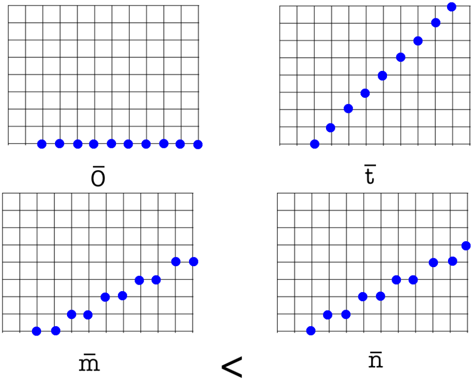
Perversities

(1) A classical perversity $\overline{p}$ is a sequence of integers

$$\overline{p}(1), \overline{p}(2), \ldots.$$

(2) A GM perversity is a classical perversity such that :
$\overline{p}(1) = \overline{p}(2) = 0$ and $\overline{p}(i) \leq \overline{p}(i+1) \leq \overline{p}(i) + 1$.
(3) Perversities form a poset $\mathcal{P}erv$. $\overline{p} \to \overline{q}$ if $\overline{p} \leq \overline{q}$.
(4) We can add perversities.

Examples



Dual perversities : $\overline{p} + \overline{q} = \overline{t}$, we set $D\overline{p} = \overline{t} - \overline{p}$.

Perverse objects

Definition :

Let **C** be a category a perverse object in **C** is a functor

$$M_\bullet : \mathcal{P}erv \to \mathbf{C}$$

If **C** is monoidal symmetric we define perverse monoids (commutative) :

$$\mu : M_{\overline{p}} \Box M_{\overline{q}} \to M_{\overline{p}+\overline{q}}.$$

Homotopical algebra of perverse objects

Theorem (M. Hovey)

The category of perverse rational CDGA's (commutative monoids in perverse cochain complexes) is a Quillen model category.
The category of perverse E_∞-dg cochain algebras is a Quillen model category.

Intersection homology

M. Goresky and R. MacPherson, and also H. King, introduced an intersection chain complex :

$$C_*^{\overline{p}}(X;R) \subset C_*(X;R).$$

Each singular chain has a perverse degree that controls the way that it intersects each strata this perverse degree is required to be bounded by $\overline{p}$. We set :

$$H_*^{\overline{p}}(X;R) = H_*(C_*^{\overline{p}}(X;R)).$$

Properties of $H^\bullet_*$

(1) Intersection cohomology is a functor

$$H^\bullet_* : \mathbf{Strat_n} \to \mathcal{P}erv_n - gRmod.$$

(2) It satisfies Mayer-Vietoris.
(3) $H^\bullet_*(X \times \mathbb{R}; R) \cong H^\bullet_*(X; R)$.
(4) If M is manifold of dimension $n-1$ then we have :
- $H^{\overline{p}}_k(cM; R) \cong H_k(M; R)$ when $k \leq n-1-\overline{p}(n)$,
- $H^{\overline{p}}_k(cM; R) \cong 0$ when $k > n-1-\overline{p}(n)$.
(5) If W is a manifold of dimension n then we have :

$$H^{\overline{p}}_k(W; R) \cong H_k(W; R).$$

Intersection cohomology

For a pseudomanifold there are two ways to define intersection cohomology :
(1) Linear dual :

$$H^*_{\overline{p}}(X;R) = H^*(Hom(C_*^{D\overline{p}}(X;R),R))$$

(2) Sheafification : using Borel-Moore theory we get a sheaf $\mathbf{IC}^*_{\overline{p}}$.
These two versions coincide when X is oriented and R is a field but not in general !

Thom-Whitney cochains

Theorem 1. (D. C., M. Saralegui, D. Tanré)

We have a functor :

$$\widetilde{N}^*_\bullet : \mathbf{Strat_n}^{op} \to \mathcal{P}erv_n - E_\infty dgas.$$

Such that $H^*(\widetilde{N}^*_\bullet(X))$ is isomorphic to $H^*_\bullet(X;R)$ when R is a field.

The case of one isolated singularity

Let $X = M \cup_{\partial M} c(\partial M)$, then

$$\widetilde{N}^*_\bullet(X;R) \simeq C^*(M) \oplus_{C^*(\partial M;R)} \tau^{\leq \bullet(dim(M))} C^*(\partial M;R)$$

where

$$\tau^{\leq k}(C^*)^i = \begin{cases} C^i, i < k, \\ Ker(d^i : C^i \to C^{i+1}), i = k, \\ 0, i > k. \end{cases}$$

The case of one isolated singularity II

If X has one isolated singularity then :

$$H^*(\widetilde{N}^*_\bullet(X;R)) = \begin{cases} H^i(M;R), i \leq k, \\ Ker(H^i(M;R) \to H^i(\partial M;R)), i = k+1, \\ H^i(X;R), i > k+1. \end{cases}$$

Thom-Whitney cochains 2.

Theorem 2. (D. C., M. Saralegui, D. Tanré)

We have a functor :

$$\widetilde{A_{PL}}^*_\bullet : \mathbf{Strat_n}^{op} \to \mathcal{P}erv_n - cdgas.$$

Such that $H^*(\widetilde{A_{PL}}^*_\bullet(X))$ is isomorphic to rational intersection cohomology.

Thom-Whitney cochains 3.

Theorem 3. (D. C., M. Saralegui, D. Tanré)

Functors $\widetilde{A_{PL\bullet}}^*$ and $\widetilde{N}^*_\bullet$ factors trough a combanitorial category of decomposed Δ-sets : $\mathbf{F}\Delta$.

$$\widetilde{N}^*_\bullet : \mathbf{Strat}_n \xrightarrow{Sing^F} \mathbf{F}\Delta \xrightarrow{N^*_\bullet} \mathcal{P}erv_n - E_\infty dgas.$$

Moreover there exist perverse Eilenberg Mac-Lane $\mathbf{F}\Delta$-sets $K_\bullet(R,k)$ such that

$$H^k(\widetilde{N}^*_\bullet(X)) \cong [Sing^F(X), K_\bullet(R,k)]_{\mathbf{F}\Delta}.$$

Remarks

(1) We have used the E_∞-structure to define and study steenrod operations in intersection cohomology answering to a conjecture of M. Goresky and W. Pardon.
(2) We have also developped a rational intersection homotopy theory with a perverse minimal theory and get new topological invariants.
(3) We can speak of intersection formality.
(4) Intersection homotopy groups and generalized intersection cohomology theories have to be defined. The algebra of additive cohomology operations has to be determined.

Formality

Theorem 4. (D. C., J. Cirici)

Complex projective hypersurfaces with isolated singularities are formal.

Poincaré duality.

Theorem 5. (D. C., M. Saralegui, D. Tanré)

Let X be a compact oriented pseudomanifold of n, we get a commutative diagram

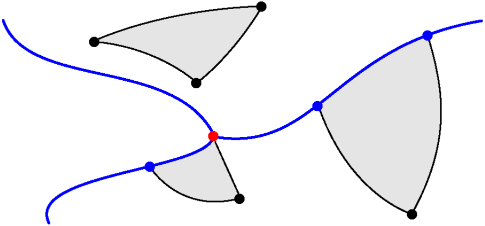
$$\begin{array}{ccc} C^*(X;R) & \xrightarrow{-\cap[X]} & C_{n-*}(X;R) \\ \downarrow & & \uparrow \\ \widetilde{N}^*_{\overline{p}}(X;R) & \xrightarrow{-\cap[X]} & C^{\overline{p}}_{n-*}(X;R) \end{array}$$

where the bottom cap product is a quasi-isomorphism.

Remarks

(1) $C^{\bullet}_{*}(X;R)$ is a left-module over $\widetilde{N}^{*}_{\bullet}(X;R)$.
(2) Over an oriented pseudomanifold the sheafification of the $\widetilde{N}^{*}_{\bullet}(-;R)$ is isomorphic to $\mathbf{IC}^{*}_{\bullet}$ in the derived category $D(X)$.

Filtered simplices



$\sigma \cap X_i$ must be a face

Perverse degree : $deg(\sigma)_i = dim(\sigma \cap X_{n-i})$.

Perverse chains

- A simplex is $\overline{p}$-admissible if $deg(\sigma)_i \leq dim(\sigma) - i + \overline{p}(i)$,
- A singular chain is $\overline{p}$-perverse : if c is a sum of $\overline{p}$-admissible simplexes and also ∂c.
- complex : $C_*^{\overline{p}}(X, R) \subset C_*(X; R)$.

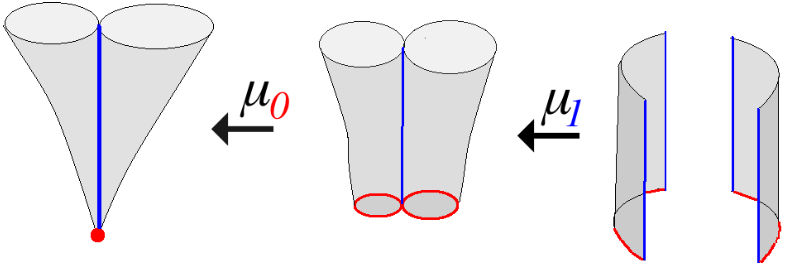
Filtered simplices II

Any filtered simplexe $\sigma : \Delta \to X$ admits a join decomposition :

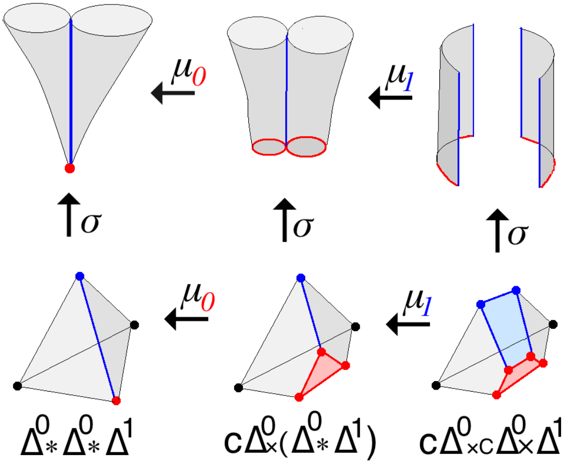
$$\Delta_0 * \cdots * \Delta_i * \cdots * \Delta_n$$

where $\Delta_0 * \cdots * \Delta_i = \sigma^{-1}(X_j)$.
We define a category Δ^F of filtered simplex : objects are joins of Δ^k morphisms are faces.
We consider the category $\mathbf{F}\Delta$ of functors $(\Delta^F)^{op} \to \mathit{Sets}$.
We get the functor Sing^F : $\mathbf{Strat} \to \mathbf{F}\Delta$ of filtered singular simplices.

Topological unfolding



Simplical unfolding



Hidden faces

The simplicial unfolding associates to each filtered simplex

$$\Delta_0 * \Delta_1 * \cdots * \Delta_i * \cdots * \Delta_n$$

a prismatic set

$$c\Delta_0 \times c\Delta_1 \times \cdots \times c\Delta_i \times \cdots \times \Delta_n$$

this prismatic set has a number of additional faces called hidden faces wich are "parallel" to the singular strata.

Local constructions

In order to define a funtor

$$\widetilde{N}^*_\bullet : \mathbf{F}\Delta \to \mathcal{P}erv_n - E_\infty dgas$$

it is sufficient to restrict to the category of decomposed simplices Δ^F. Thus we just have to give the value of the functor on each

$$\Delta_0 * \Delta_1 * \cdots * \Delta_i * \cdots * \Delta_n.$$

Local constructions II

To each $\Delta_0 * \Delta_1 * \cdots * \Delta_i * \cdots * \Delta_n$ we consider the cochain algebra

$$C^*(c\Delta_0; R) \otimes C^*(c\Delta_1; R) \otimes \cdots \otimes C^*(\Delta_n; R)$$

we view it as a cochain on the simplicial unfolding.

Local constructions III

Definition

Let $\omega = a_0 \otimes a_1 \otimes \cdots \otimes a_n$ the i-**th vertical degree** of ω is equal to

$$vert_i(\omega) = |a_{n-i+1} \otimes \cdots \otimes a_n|$$

Let $\eta \in C^*(c\Delta_0; R) \otimes C^*(c\Delta_1; R) \otimes \cdots \otimes C^*(\Delta_n; R)$ the i-**th perverse degree** of η denoted by $Pvert_i(\eta)$ is the i-th vertical degree of the restriction of η to the hidden face.
Then η is $\overline{p}$-admissible if $Pvert_i(\eta) \leq \overline{p}(i)$ for any i. Finally we get a cochain complex

$$\widetilde{N}^*_{\overline{p}}(\Delta_0 * \ldots \Delta_n; R) \subset C^*(c\Delta_0; R) \otimes \cdots \otimes C^*(\Delta_n; R)$$

The End

Thank you !