Kahn’s realizability problem

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Realizability. How to play

- Give you a (abstract) group $G$
- Give you category $C$
- Give me back an object $X$ in $C$ such that $\text{Aut}_C(X) \cong G$

**Example 1**
- $G = \mathbb{Z}_2$
- $C = \text{HoTop}_*$
- Then, $X = S^n$

**Example 2**
- $G = \mathbb{Z}_p$, $p$ odd
- $C = \text{Groups}$
- Then, $\text{Aut}_C(X) \not\cong \mathbb{Z}_p$, $\forall X$

So, finite groups can not, in general, be realized in the category of groups

Are finite groups realizable in $\text{HoTop}_*$?
Our problem

Let $\mathcal{E}(X) =$ group of homotopy classes of self homotopy-equivalences of $X$

finite group $G$

$\Downarrow \text{Realization}$

$G \cong \mathcal{E}(X)$ for some $X$?
Overview

• Proposed by Kahn in the late 60’s, appears recurrently in literature
• The only general known procedure to tackle this problem is when $G = Aut(\pi)$, $\pi$ a group. Then $X = K(\pi, n)$, since $\mathcal{E}(X) \cong Aut(\pi)$.
• Approach $\mathcal{E}(X)$ by its distinguished subgroups

\[ \mathcal{E}_{\#}(X), \mathcal{E}_*(X), \mathcal{E}^*(X) \ldots \]

Example

\[ \mathbb{Z}_2 \cong \mathcal{E}(S^n) \]
\[ \mathbb{Z}_2 \cong \mathcal{E}(K(\mathbb{Z}_3, n)) \text{ since } Aut(\mathbb{Z}_3) \cong \mathbb{Z}_2 \]
\[ \mathbb{Z}_2 \cong \mathcal{E}(X) \text{ for some 1-connected rational space } X \] [Arkowitz-Lupton’00]

Which finite groups are realizable by simply connected rational spaces?
New perspective

Idea
Introduce graphs on the picture

\[ \text{groups} \rightarrow \text{graphs} \]
\[ \text{graphs} \rightarrow \text{DGA’s} \]
\[ \text{DGA’s} \rightarrow \text{rational homotopy types} \]

**Theorem (Frucht’39, Realizability in } \mathcal{C} = \text{Graphs} \)**
Every finite group \(G\) is realizable by a finite, connected and simple graph \(G\).

**Example 1** (\(G = \mathbb{Z}_3\), Cayley graph \(\rightarrow\) simple graph)
New perspective

Example 2 ($G = \Sigma_4$, Cayley graph $\rightarrow$ simple graph )

$\alpha = (1, 2)$

$\beta = (2, 3)$

$\gamma = (3, 4)$
New perspective

Example 2 ($G = \Sigma_4$, Cayley graph $\rightarrow$ simple graph )
New perspective

Example 2 ($G = \Sigma_4$, Cayley graph $\rightarrow$ simple graph )
Problem 1

Let $G = (V, E)$ be a finite, simple, connected graph (with more than one vertex). Does there exist a space $X$ such that $\text{Aut}(G) \cong \mathcal{E}(X)$?
Solving Problem 1

▷ First, restrict ourselves $Graph_{fm} \subset Graph$.
▷ Then, construct

$$A : Graph_{fm} \longrightarrow DGA$$

$$(A_g, d) = (\Lambda(x_1, x_2, y_1, y_2, y_3, z) \otimes \Lambda(x_v, z_v | v \in V), d)$$

- generators in dimensions: $|x_1| = 8, |x_2| = 10, |y_1| = 33, |y_2| = 35, |y_3| = 37, |z| = 119, |x_v| = 40, |z_v| = 119$,

- differentials:

$$
\begin{align*}
  d(x_1) &= 0 & d(y_3) &= x_1 x_2^3 \\
  d(x_2) &= 0 & d(x_v) &= 0 \\
  d(y_1) &= x_1^3 x_2 & d(z) &= y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15} + x_2^{12} \\
  d(y_2) &= x_1^2 x_2^2 & d(z_v) &= x_v^3 + \sum_{[v,w] \in E} x_v x_w x_2^4
\end{align*}
$$

- $A$ is contravariant (morphisms are as expected).
Solving Problem 1

First, restrict ourselves $\text{Graph}_{fm} \subset \text{Graph}$.

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Homotopically Rigid

Encodes $\mathcal{G}$

- generators in dimensions: $|x_1| = 8, |x_2| = 10, |y_1| = 33, |y_2| = 35, |y_3| = 37, |z| = 119, |x_v| = 40, |z_v| = 119,

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- Homotopically Rigid
- Encodes $G$

• generators in dimensions: $|x_1| = 8, |x_2| = 10, |y_1| = 33, |y_2| = 35, |y_3| = 37, |z| = 119, |x_v| = 40, |z_v| = 119,$

• differentials:

$$d(x_1) = 0 \quad d(y_3) = x_1 x_2^3$$
$$d(x_2) = 0 \quad d(x_v) = 0$$
$$d(y_1) = x_1^3 x_2 \quad d(z) = y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15} + x_2^{12}$$
$$d(y_2) = x_1^2 x_2^2 \quad d(z_v) = x_v^3 + \sum_{[v, w] \in E} x_v x_w(u_1 x_1^5 + u_2 x_2^4), u_1, u_2 \in \mathbb{Q}^*$$

• $A$ is contravariant (morphisms are as expected).
Solving Problem 1

Theorem
Let $\mathcal{G}, A_\mathcal{G}$ defined as previously. Then:

- There exists a split short exact sequence

$$K \rightarrow \text{Aut}(A_\mathcal{G}) \rightarrow \text{Aut}(\mathcal{G})$$

where $K$ is abelian and torsion-free.

- $A_\mathcal{G}$ is an elliptic algebra (hence Poincaré duality) of formal dimension $d = 208 + 80|V|$

- Let $X_\mathcal{G}$ the rational elliptic 1-connected space whose Sullivan minimal model is $A_\mathcal{G}$. The monoid of self-homotopy classes of $X_\mathcal{G}$ is

$$[X_\mathcal{G}, X_\mathcal{G}] = \{f_0, f_1\} \cup \text{Aut}(\mathcal{G})$$
Solving Problem 1

**Theorem**
Every finite group $G$ is realized by infinitely many (non homotopically equivalent) rational elliptic spaces $X$. That is, $G \cong \mathcal{E}(X)$.

Before we get into specific categories of problems, let me say that there are two very broad problems - somewhat vague and general - that most workers agree are very important:

**A.** Calculate the groups $\mathcal{E}(X)$ explicitly in as many cases as possible, and express the known calculations in the most simple and concrete terms.

**B.** Develop applications of the group $\mathcal{E}(X)$ to other parts of topology (and mathematics in general).

D. Kahn’90
Applications

Idea (Crowley-Löh, 2015)
Degree theorems “à la Gromov” are strongly related with the existence of inflexible manifolds

Definition (Inflexible manifold)
An oriented closed connected manifold $M$ is inflexible if
\[
\{\deg f \mid f : M \to M \text{ continuous}\} \subset \{-1, 0, 1\}
\]

Inflexible manifolds are constructed (using rational homotopy theory) in dimensions $64 \cup \{d \cdot k \mid k \in \mathbb{N}, d = 108, 208, 228\} \equiv 0 \pmod{4})$. 
Applications

Recall that

• $X_G$ is an elliptic space of formal dimension $d = 208 + 80|V|$ such that

$$[X_G, X_G] = \{f_0, f_1\} \cup \text{Aut}(G)$$

• Therefore, if $X_G$ is the rationalisation of a manifold $M$, then $M$ is inflexible

But $d \equiv 0 \pmod{4}$ so we are in the bad range of the obstruction theory of Barge and Sullivan

Modifying our construction we get . . .
Applications

Theorem
For any connected finite graph $G$, there exist $\tilde{A}_G, \tilde{X}_G$ such that:

- $\tilde{A}_G$ is an elliptic dga of formal dimension $d = 2(208 + 80|V|) - 1$. Since $d \equiv 3 \pmod{4}$, $\tilde{X}_G$ is the rationalization of a $d$-manifold $M_G$.
- The self-monoid $[\tilde{X}_G, \tilde{X}_G] \cong \{f_0, f_1\} \cup \text{Aut}(G)$. Hence $M_G$ is inflexible.

Theorem
For every finite group $G$, there exist infinitely many inflexible manifolds $M_G$ such that

$$\mathcal{E}((M_G)_\mathbb{Q}) \cong G$$
What happens if $G$ acts on a module $M$?
Realizability level 2. How to play

• Algebraic structure \((G, M)\)
  
  \(G\) is a group, \(M\) is a finitely generated \(\mathbb{Z}G\)-module

• Homotopy invariant \((\mathcal{E}(-), \pi_k(-))\)
  
  \(\pi_k(-)\) is a \(\mathbb{Z}\mathcal{E}(-)\)-module

Problem 2 (realizability of actions)
Is there a finite Postnikov piece \(X\) such that the \(\mathbb{Z}G\)-module \(M\) is isomorphic to the \(\mathbb{Z}\mathcal{E}(X)\)-module \(\pi_k(X)\), for some \(k \geq 2\)?

▷ “Homotopique dual” of the \(G\)-Moore spaces problem (Steenrod’60)

▷ It implies realizability of groups
Realizability level 2. How to play

- Algebraic structure \((G, V)\)
  \(G\) is a group, \(V\) is a finitely generated \(\mathbb{Q}G\)-module

- Homotopy invariant \((\mathcal{E}(-), \pi_k(-))\)
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▷ “Homotopique dual” of the \(G\)-Moore spaces problem (Steenrod’60)
▷ It implies realizability of groups
New Perspective

Idea
Introduce Invariant Theory on the picture.

- $G$ acts on $\mathbb{Q}[V]$: for $g \in G$, $p \in \mathbb{Q}[V]$, $(gp)(v) = p(g^{-1}v)$.
- $G$-invariant function: $p \in \mathbb{Q}[V]$ such that for all $g \in G$, $gp = p$.
- The invariant ring $\mathbb{Q}[V]^G$: all the $G$-invariant functions in $\mathbb{Q}[V]$

( Characterization of finite $G \leq \text{GL}(V)$, Hilbert, Noether)

Let $V$ be a finitely generated and faithful $\mathbb{Q}G$-module. Then, there exists algebraic forms $p_1, \ldots, p_r \in \mathbb{Q}[V]^G$ such that, for $f \in \text{GL}(V)$

$$f \in G \text{ if and only if } p_i \circ f = p_i, \forall i$$

we modify those algebraic forms
Solving Problem 2

Lemma
There exist a family \( Q = \{q_0, q_1, \ldots q_r, q_{r+1}\} \subset \mathbb{Q}[V]^G \) where

1. \( q_0 = \sum_{1 \leq j \leq N} \lambda_j v_j^2 \), for a good choice of basis of \( V^* \) (\( N = \text{dim}_\mathbb{Q} V \)),
2. \( \deg(q_i) < \deg(q_{i+1}) \) for all \( i \),
3. \( q_{r+1} = (q_0)^s \) for \( s \gg N \)

such that \( G \) is the orthogonal group \( O(Q) \leq GL(V) \).

Definition (Realizable family of forms)
A family of algebraic forms \( Q \subset \mathbb{Q}[v_1, \ldots, v_N] \) verifying 1, 2 and 3.

For an arbitrary realizable family, and any \( n > \deg(q_{r+1}) \) …
Solving Problem 2

\[ M_{(Q,n)} = \left( \Lambda(x_1, x_2, y_1, y_2, y_3, z, v_j \mid j = 1, \ldots, N), d \right) \]

\[ \deg x_1 = 8, \quad d(x_1) = 0 \]
\[ \deg x_2 = 10, \quad d(x_2) = 0 \]
\[ \deg y_1 = 33, \quad d(y_1) = x_1^3x_2 \]
\[ \deg y_2 = 35, \quad d(y_2) = x_1^2x_2^2 \]
\[ \deg y_3 = 37, \quad d(y_3) = x_1x_2^3 \]
\[ \deg v_j = 40, \quad d(v_j) = 0 \]
\[ \deg z = 80n + 39, \quad d(z) = \sum_{i=1}^{r+1} q_i x_1^{10n+5-5 \deg(q_i)} + q_0(x_1^{10n-5} + x_2^{8n-4}) \]
\[ + x_1^{10(n-1)}(y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6) \]
\[ + x_1^{10n+5} + x_2^{8n+4}. \]
Solving Problem 2

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& & & + x_1^{10(n-1)} (y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6) \\
& & & + x_1^{10n+5} + x_2^{8n+4}.
\end{align*} \]

Codifies the action
Solving Problem 2

Theorem

\[ \mathcal{E}(\mathcal{M}(Q, n)) \cong O(Q) \]

Corollary

Let \( G \) be a finite group, and \( V \) a finitely generated faithful \( \mathbb{Q}G \)-module. Then, there exists a Postnikov piece \( X \) such that, for some \( k \geq 2 \),

\[ (G, V) \cong (\mathcal{E}(X), \pi_k X) \]

Example (realization of infinite groups)

Let \( O(m; k) < GL_{m+k}(\mathbb{R}) \) preserving:

\[ q_0 = x_1^2 + x_2^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_{m+k}^2. \]

The family \( Q = \{ q_0, (q_0)^{m+k+1} \} \subset \mathbb{Q}[x_1, \ldots, x_{m+k}] \) is realizable. Then,

\[ O(Q) \] can be realized by infinitely many (rational) spaces.

\[ O(Q) \cong O(m; k)(\mathbb{Q}), \] which is an infinite group for \( m \geq 2 \).
Our solutions to Problem 1 and Problem 2 depend on:

- A very specific homotopically rigid algebra. It is not unique:

For a fixed $k > 4$, define $\mathcal{M}_k = \left( \Lambda(x_1, x_2, y_1, y_2, y_3, z), d \right)$

\[
\begin{align*}
\deg x_1 &= 5k - 2, & d(x_1) &= 0 \\
\deg x_2 &= 6k - 2, & d(x_2) &= 0 \\
\deg y_1 &= 21k - 9, & d(y_1) &= x_1^3x_2 \\
\deg y_2 &= 22k - 9, & d(y_2) &= x_1^2x_2^2 \\
\deg y_3 &= 23k - 9, & d(y_3) &= x_1x_2^3 \\
\deg z &= 15k^2 - 11k + 1, & d(z) &= x_1^{3k-12}(x_1^2y_2y_3 - x_1x_2y_1y_3 + x_2^2y_1y_2) \\
& & & + x_1^{\frac{6k-2}{2}} + x_2^{\frac{5k-2}{2}}.
\end{align*}
\]

**Theorem** $[\mathcal{M}_k, \mathcal{M}_k] = \{0, 1\}$

- Rational homotopy theory (finite type over $\mathbb{Q}$, not over $\mathbb{Z}$).
Realizability. An integral approach

Following our approach for $\mathbb{Q}$

- Find an integral homotopically rigid space.
- Find a functor from a combinatorial category to integral spaces.

Idea
Introduce Toric Topology in the picture
Homotopically rigid space

\[ \mathbb{H}P^\infty \simeq BS^3 \]

**Definition (Degree)**

For \( f : \mathbb{H}P^\infty \to \mathbb{H}P^\infty \), if \( \deg(\Omega f : S^3 \to S^3) = k \), we say that \( \deg(f) = k \).

(Feder-Gitler, Sullivan)

Self-maps of \( \mathbb{H}P^\infty \) have either degree zero or any odd square integer.

(Classification Theorem, Mislin)

Self-maps of \( \mathbb{H}P^\infty \) are classified up to homotopy by their degree.

**Corollary**

\[ \mathcal{E}(\mathbb{H}P^\infty) = \{1\} \]
Polyhedral product functor

Let $K$ be a simplicial complex on a set $V$ of vertices, $v_1, \ldots, v_n$. Let $(X, \ast)$ be a pointed space.

Definition (Buchstaber-Panov, Bahri-Bendersky-Cohen-Gitler, Notbohm-Ray)

For $\sigma \subset V$ face of $K$, the $\sigma$—power of $X$ is:

$$X^\sigma = \{(x_1, \ldots, x_n) \in X^n \mid x_i = \ast \text{ if } v_i \notin \sigma\}$$

The polyhedral product is the (homotopy) colimit of the diagram:

$$X^K : CAT(K) \to \text{Top}_\ast$$

$$\sigma \mapsto X^\sigma$$

By abuse of notation, we will also denote by $X^K$:

$$\text{hocolim} X^K \simeq \text{colim} X^K = \bigcup_{\sigma \in K} X^\sigma \subseteq X^n$$
Polyhedral product functor, examples

Example 1

\[ X^{\Delta[n-1]} \cong X^n \] the n-fold product

\[ X^{\partial \Delta[n-1]} \cong T^n X \] the fat wedge

\[ X^\emptyset \cong * \] the trivial space

Example 2 (Davis-Januszkiewicz space)
For \( X = BS^1 \), \( (BS^1)^K \cong DJ(K) \) where \( H^*(DJ(K); \mathbb{Z}) \cong \mathbb{Z}[K] \).

Recall that: \( \mathbb{Z}[K] = S_{\mathbb{Z}}(V)/(v_U : U \notin K) \).

Kahn’s realizability problem
Conjecture

For a simplicial complex $K$,

$$\mathcal{E}((BS^3)^K) \cong Aut(K)$$

Example 1
For $K = \Delta[n - 1]$

$$\mathcal{E}((BS^3)^n) \cong \Sigma_n$$ (Iwase)

Example 2
For $X = BS^1$, $K = \Delta[n - 1]$

$$\mathcal{E}((BS^1)^\Delta[n^{-1}]) \cong GL(n, \mathbb{Z}) \not\cong \Sigma_n \cong Aut(\Delta[n - 1])$$
Solving Conjecture

Let $K$ be a simplicial complex

**Proposition**

$$\mathcal{E}((BS^3)^K)/\mathcal{E}^*((BS^3)^K) \cong Aut(K)$$

**Proof**

▷ First, show $H^*((BS^3)^K; \mathbb{Z}) \cong \mathbb{Z}[K]$ with generators in degree 4.

▷ Then, identify $\mathcal{E}((BS^3)^K)/\mathcal{E}^*((BS^3)^K)$ to the image of

$$\psi : \mathcal{E}((BS^3)^K) \to Aut(H^4((BS^3)^K; \mathbb{Z}))$$

$$f \mapsto H^4(f; \mathbb{Z})$$

▷ Finally, the entries of $M_f \in GL(n, \mathbb{Z})$ induced by $H^4(f; \mathbb{Z})$ are non-negative integers (degrees of self-maps of $BS^3$). Then $M_f$ and $M_{f-1}$ are permutation matrices, and $Im \psi = Aut(K)$.

□
Solving Conjecture

Theorem
Let $K$ be a simplicial complex of dimension 1. Then

$$\mathcal{E}^*\left((BS^3)^K\right) \cong \{1\}$$

Proof (techniques of Dwyer-Mislin, Jackowski-McClure-Oliver, Notbom-Ray)
Fix notation $X = BS^3$.

$\triangleright$ Step 1 We have:

$$[X^K, X^K] \xrightarrow{\text{injection}} [X^K, X^n] \xrightarrow{\{\pi_i\}_1^n} [X^K, X] \xrightarrow{\text{injection}} \prod_p [X^K, X_p^\wedge]$$

$f \xrightarrow{\text{injection}} f \xrightarrow{\{f_j\}_1^n} \{f_j^\wedge | p\}_1^n$

we also have, for a face $\sigma$ of $K$:

$$[X^\sigma, X] \cong \{(0, 0, \ldots, a_i, 0) | a_i = 0 \text{ or } a_i \text{ odd square}\}$$

$\text{dim} \sigma + 1$
Solving Conjecture

- **Step 2** We then have, for every \( j = 1, \ldots, n \), for every \( p \) prime:

\[
E^*(X^K) \xrightarrow{\sim} \{ [X^\sigma, X^\wedge_p] \mid \sigma \in CAT(K) \}
\]

\[
f \xrightarrow{\sim} f_j^\sigma \xrightarrow{\sim}_p \begin{cases} 
  \pi_j & \text{if } v_j \in \sigma \\
  * & \text{if } v_j \notin \sigma 
\end{cases}
\]

Is there \( f \neq Id_{X^K} \) inducing the same family?

- **Step 3** The obstruction for the unicity lies in \( \lim^i \Pi_i^p \) for

\[
\Pi_i^p : CAT^{op}(K) \to Ab
\]

\[
\sigma \mapsto \pi_i(map(X^\sigma, X^\wedge_p)_{f_j^\sigma})
\]

that can be computed as the cohomology of a cochain complex

\[
N^n(\Pi_i^p) = \prod_{\sigma_0 \to \sigma_1 \to \cdots \to \sigma_n} \Pi_i^p(\sigma_n)
\]

As \( \dim K = 1 \), \( N^{\geq 3}(\Pi_1^p) = 0 \), \( N^2(\Pi_2^p) = 0 \), and \( H^1(N^*(\Pi_1^p)) = 0 \). □
Solving Conjecture

Corollary 1
Let $K$ be a simplicial complex of dimension 1. Then

$$\mathcal{E}((BS^3)^K) \cong Aut(K)$$

Corollary 2
Every finite group is realizable by infinitely many integral spaces.