Equivariant calculus and the tower of the identity on pointed G-spaces

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Overview

1. Issues With Calculus

2. Equivariant Excision and the Equivariant Taylor Tower

3. The Layers of the Tower

4. Indexed Symmetric Powers and the Layers for the Identity
Calculus of Functors \((G = 1)\)

Let \(F : \mathcal{C} \to \mathcal{D}\) be a homotopy functor between model categories.

**Theorem (Goodwillie)**

There is a “Taylor tower” of functors

\[
\cdots \to P_n F \to P_{n-1} F \to \cdots \to P_2 F \to P_1 F \to F(*)
\]

which satisfies:

- \(F(X) \simeq \operatorname{holim}_n P_n F(X)\), sometimes,
- \(P_n F\) is “\(n\)-excisive” (a homology theory when \(n = 1\)),
- For \(\mathcal{C} = \mathcal{D} = \text{Top}_x\) the layer \(D_n F = \operatorname{hofib}(P_n F \to P_{n-1} F)\) decomposes as:

\[
D_n F(X) \simeq \Omega^\infty (\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}
\]

Where \(\partial_n F\) is a spectrum with \(\Sigma_n\)-action (naïve).

This is “Brown representability” for reduced homology theories of degree \(n\).
What Goes Wrong Equivariantly?

Let $G$ be a finite group. Let $\text{Top}^G_*$ be the model category of $G$-spaces and fixed-points equivalences:

**Definition**

$f: X \to Y$ is a w.e. if $f^H: X^H \to Y^H$ is a w.e. of spaces for all $H \leq G$.

We can of course set $\mathcal{C} = \mathcal{D} = \text{Top}^G_*$ and take the tower of $F: \text{Top}^G_* \to \text{Top}^G_*$. However:

**Issues**

- The layer is a naïve infinite loop space

$$D_nF(X) \simeq \Omega^\infty (\partial_nF \wedge X^\wedge n)_{h\Sigma_n}$$

$(\partial_nF$ is a naïve $G \times \Sigma_n$-spectrum).

- This decomposition holds only when the $G$-action on $X$ is trivial.
The Case $n = 1$ (Blumberg)

Let $F : \text{Top}^G_\ast \to \text{Top}^G_\ast$ be reduced: $F(*) \simeq \ast$. Then

$$P_1 F(X) \simeq \hocolim_{n \in \mathbb{N}} \Omega^n F(\Sigma^n X)$$

**Construction**

$$P_G F(X) := \hocolim_{n \in \mathbb{N}} \Omega^{n \rho_G} F(\Sigma^{n \rho_G} X)$$

where $\rho_G = \mathbb{R}[G]$ is the regular representation of $G$.

**Theorem (Blumberg)**

$P_G F(X)$ is the universal "$G$-linear" approximation of $F$: $P_G F$ is linear and

$$P_G F(\bigvee J X) \simeq \prod_j P_G F(X)$$

for every finite $G$-set $J$.

It follows that "$G$-linear functors are equivalent to $G$-spectra".
Higher Equivariant Excision

Program

1. Formulate $G$-excision in “cubical terms”,

2. Extend this notion to $J$-excision, for finite $G$-sets $J$,

3. Extend the framework from $\text{Top}_G^*$ to general “equivariant homotopy theories” (e.g. $G$-spectra).
Let $G$ be a finite group.

**Definition (D-Moi/Hill)**

A $G$-model category is a functor $\mathcal{C}: \mathcal{O}_G^{op} \to \text{ModCat}$ where:

- $\mathcal{O}_G = \{\text{transitive } G\text{-sets and } G\text{-maps}\}$ is the orbit category of $G$,
- $\text{ModCat}$ is the category of model categories and left and right Quillen functors.

We will further assume that:

- $\mathcal{C}(G/H) = \mathcal{C}^H$ is the category of $H$-objects in some category $\mathcal{C}$ (as 1-categories),
- The functors $\mathcal{C}^H \to \mathcal{C}^K$ are the standard restrictions and conjugations.

This is a homotopy theory “parametrized” by the orbit category of $G$.

[Barwick-D-Glasman-Nardin-Shah] for an $\infty$-categorical setting.

**Example)**

- The categories $\text{Top}^H$ with the fixed-points model structures,
- The categories $\text{Sp}^H$ of orthogonal $H$-spectra with the $H$-stable model structures.
Equivariant Diagrams

Let $G$ be a finite group, $I$ a category with $G$-action and $\mathcal{C}$ a $G$-model category.

**Theorem (D-Moi)**

There exists a model category of $I$-shaped diagrams $X : I \to \mathcal{C}$ with “$G$-action”: natural maps $g : X_i \longrightarrow X_{gi}$ compatible with the group structure.

Example) Let $G = \mathbb{Z}/2$, and $I = (\bullet \to \bullet \leftarrow \bullet)$ with $G$-action

\[
\bullet \quad \downarrow \quad \bullet \quad \leftarrow \quad \bullet
\]

If $Y$ is a pointed $\mathbb{Z}/2$-space, the following is a $\mathbb{Z}/2$-equivariant diagram in $\text{Top}_*$:

\[
\ast \quad \downarrow \quad Y
\]
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Equivariant Homotopy Limits and Colimits

Let $G$ be a finite group, $I$ a category with $G$-action and $\mathcal{C}$ a $G$-model category.

**Theorem (D-Moi)**

There are well-behaved homotopy limit and colimit functors

$$\text{holim}, \text{hocolim}: \{I\text{-shaped } G\text{-diagrams in } \mathcal{C}\} \to \mathcal{C}^G$$

Example) Let $G = \mathbb{Z}/2$ and $Y$ a pointed $\mathbb{Z}/2$-space, then

$$\text{holim} \begin{pmatrix}
  * \\
  \downarrow \\
  Y
\end{pmatrix} = \Omega Y = Map_*(S^1, Y)$$
Equivariant Homotopy Limits and Colimits

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Example) Let $G = \mathbb{Z}/2$ and $Y$ a pointed $\mathbb{Z}/2$-space, then

$$\text{holim} \begin{pmatrix} * \\ * \rightarrow Y \\ \downarrow \\ * \rightarrow \end{pmatrix} = \Omega^\text{sign} Y = Map_*(S^\text{sign}, Y)$$

**Consequence**

This gives a systematic way of incorporating representations into equivariant homotopy theory.
Reformulation of $G$-Excision

Let $J$ be a finite $G$-set, and $\mathcal{P}(J)$ the category of (all) subsets of $J$. $G$ acts on $\mathcal{P}(J)$ by $g \cdot U = \{g \cdot j \mid j \in U\}$.

Definition (Equivariant cubes)

A $J$-cube is a diagram $X: \mathcal{P}(J) \to \mathcal{C}$ with a $G$-action.

Let $F: \mathcal{C}^G \to \mathcal{D}^G$ be a homotopy functor.

Definition ($G$-excision)

$F$ is $G$-excisive if

$$F_*: \{G_+\text{-cubes in } \mathcal{C}\} \longrightarrow \{G_+\text{-cubes in } \mathcal{D}\}$$

sends cocartesian cubes to cartesian cubes. (Here $G_+ = G \cup \{+\}$.)
Reformulation of $G$-Excision

**Theorem (D-Moi)**

Suppose that $F(*) \simeq *$. The following are equivalent:

- $F$ is $G$-excisive,
- $F$ sends cocartesian $J$-cubes to cartesian $J$-cubes, for every finite $G$-set $J$,
- $F(X) \simeq \Omega^G F(\Sigma^G X)$, (that is $F \simeq P_G F$),
- $F$ is excisive and $F(\bigvee J X) \simeq \prod J F(X)$ (Blumberg’s definition).
Let $J$ be a finite $G$-set. Let $F: C^G \to D^G$ be a homotopy functor.

**Definition ($J$-excision)**

$F$ is $J$-excisive if

$$F_*: \{J_+\text{-cubes in } C\} \longrightarrow \{J_+\text{-cubes in } D\}$$

sends “strongly cocartesian” cubes to cartesian cubes.
Examples

- An $n$-excisive functor is $\underline{n}$-excisive, for the trivial $G$-set $\underline{n} = \{1, \ldots, n\}$.
- Let $M$ be a $\mathbb{Z}[G]$-module. The Dold-Thom construction $M(-) : \text{Top}_G^* \to \text{Top}_G^*$ is $G$-linear.
- Let $E$ be a $G$-spectrum. $E \wedge (-) : \text{Top}^*_G, \text{Sp}^G_G \to \text{Sp}^G_G$ is $G$-linear. In particular the identity on $\text{Sp}^G_G$ is $G$-linear.
- Let $A$ be a commutative ring, $M$ an $A$-bimodule. There is a $\mathbb{Z}/2$-spectrum

  $$\text{THR}(A; M) = \left| [k] \mapsto H M \wedge (HA)^{\underline{k}} \right|$$

  where $\mathbb{Z}/2$ acts on $\underline{k} = \{1, \ldots, k\}$ by $i \mapsto k - i + 1$. Then

  $$\text{THR}(A; M(-)) : \text{Top}_{\mathbb{Z}/2} \to \text{Sp}_{\mathbb{Z}/2}$$

  is $\mathbb{Z}/2$-excisive.
- Let $K$ be a finite $G$-set. The norm $(-)^{\underline{K}} : \text{Sp}^G \to \text{Sp}^G_G$ is $K \times G = |K| \times G$-excisive.
The Equivariant “Tower”

Let $F : \mathcal{C}^G \to \mathcal{D}^G$ be a homotopy functor.

**Theorem (D)**

There are $J$-excisive approximations $F \to P_J F$, and an essentially unique map $P_J F \to P_K F$ if there is $K \to J$ injective on orbits.

These give a diagram

\[
\begin{array}{ccc}
\vdots & \longrightarrow & \vdots \\
\downarrow & & \downarrow \\
P_n F & \longrightarrow & \cdots \longrightarrow & P_{nG} F \\
\downarrow & & & \downarrow \\
P_2 F & \longrightarrow & \cdots \longrightarrow & P_{2G} F \\
\downarrow & & & \downarrow \\
P_1 F & \longrightarrow & \cdots \longrightarrow & P_{G/K} F \\
\end{array}
\]
Basic Properties

Properties

- **Suppose** \( F(\ast) \simeq \ast \) and \( J \) transitive. Then

\[
P_J F(X) \simeq \text{hocolim}_n \Omega^n J F(\Sigma^n J X)
\]

where \( nJ \) denotes the permutation representation \( \mathbb{R}[nJ] \),

- **For every subgroup** \( H \leq G \)

\[
(P_{nG} F)|_H \simeq P_{nH} (F|_H)
\]

- **There is a non-equivariant equivalence**

\[
(P_J F)|_1 \simeq (P_{|J/G|} F)|_1
\]

We think of \( P_J \) as an enhancement of \( P_{|J/G|} \) that builds in the orbits of \( J \).
What kind of convergence should one expect? After all, often enough

\[ F \simeq \underset{n}{\text{holim}} \, P_n F \]

Consider the “naïve” and “genuine” equivariant stable homotopy monads

\[ Q = \Omega^\infty \Sigma^\infty \quad \text{and} \quad Q_G = \Omega^\infty \rho_G \Sigma^\infty \rho_G : \text{Top}_*^G \to \text{Top}_*^G \]

Arone-Kankaanrinta:

\[ \text{Tot} Q^\bullet \simeq \underset{n}{\text{holim}} \, P_n I \]

Carlsson:

\[ \text{Tot} Q^\bullet \xrightarrow{\simeq} \text{Tot} Q^\bullet_G \]

Then maybe also

\[ \underset{n}{\text{holim}} \, P_{nG} I \simeq \text{Tot} Q^\bullet_G \simeq \underset{n}{\text{holim}} \, P_n I. \]

**Theorem (D)**

Let \( J_1 \not= J_2 \not= \ldots \), and suppose \( F : \text{Top}_*^G \to \text{Top}_*^G \) commutes with fixed-points (e.g. \( F = I \)). Then

\[ \underset{n}{\text{holim}} \, P_{J_n} F \simeq \underset{n}{\text{holim}} \, P_n F \]
Delooping the Layers

By the previous result, sometimes,

\[ F \simeq \text{holim}( \ldots \rightarrow P_{nG}F \rightarrow P_{(n-1)G}F \rightarrow \ldots \rightarrow P_{2G}F \rightarrow P_{G}F ) \]

The layer \( D_{nG}F \) is \( nG \)-excisive and satisfies \( P_{kG}D_{nG}F \simeq \ast \) for \( k < n \).

**Definition**

\( \Phi \) is \( J \)-homogeneous if it is \( J \)-excisive and \( P_K \Phi \simeq \ast \) for every \( G \)-subset \( K \not\in J \).

**Theorem**

Let \( \Phi: \text{Top}_*^G \rightarrow \text{Top}_*^G \) be \( J \)-homogeneous. Then

\[ \Phi \simeq \Omega^\infty J \widehat{\Phi} \]

for some \( \widehat{\Phi}: \text{Top}_*^G \rightarrow \text{Sp}^G \). In particular

\[ D_{nG}F \simeq \Omega^\infty \rho_G \overline{D_{nG}F} \]
Digression: Equivariant Deloopings

Given $X \in \text{Top}_*^G$, how does one prove that $X \simeq \Omega^\rho G Y$?

For $G = 1$, construct a fiber sequence $X \rightarrow E \rightarrow Y$ with $E \simeq \ast$, or equivalently

$$
\begin{array}{ccc}
X & \rightarrow & E \\
\downarrow & & \downarrow \\
E & \rightarrow & Y
\end{array}
$$

homotopy cartesian, with $E \simeq \ast$.

Construction

In general, define a homotopy cartesian $G_+$-cube $Z$ with

- $Z_{\emptyset} = X$
- $Z_{G_+} = Y$
- $Z_U \simeq_{G_U} \ast$ for every $\emptyset \neq U \subseteq G_+$

Then

$$
X \simeq \text{holim}_{\emptyset \neq U \subseteq G_+} Z_U \simeq \Omega^\rho G Y
$$
Let $\Lambda$ be a finite group, $\mathcal{R}$ a collection of subgroups of $\Lambda$. Let $\overline{E\mathcal{R}}$ be a pointed $\Lambda$-space s.t.

$$(\overline{E\mathcal{R}})^\Gamma \simeq \begin{cases} S^0 & \text{if } \Gamma \in \mathcal{R} \\ \ast & \text{if } \Gamma \notin \mathcal{R} \end{cases}$$

Suppose $\Lambda = G \times \Sigma_k$, and that $\mathcal{R}$ contains only graphs of group homomorphisms $\rho: H \to \Sigma_k$, for $H \leq G$.

**Construction**

We let $(-)_{h\mathcal{R}}: \text{Sp}^{G \times \Sigma_k} \to \text{Sp}^G$ be the homotopy $\mathcal{R}$-orbits functor:

$$E_{h\mathcal{R}} := E \wedge \Sigma_k \overline{E\mathcal{R}}.$$
Let $\mathcal{F}_k$ be the collection of graph subgroups of $G \times \Sigma_k$. For $n \in \mathbb{N}$ we let

$$\mathcal{F}_k(n) = \begin{cases} 
\{ \text{graph}(\rho: H \to \Sigma_k) \mid (\rho^* k)/H = n - 1 \} & \text{if } n < k \\
\{ \text{graph}(\rho: H \to \Sigma_k) \mid (\rho^* n)/H = n - 1 \text{ or } \rho = 1 \} & \text{if } n = k \\
\emptyset & \text{if } n > k
\end{cases}$$

**Theorem (D)**

There is an equivalence of functors $\text{Sp}^G \to \text{Sp}^G$

$$D_{nG}(X^\land k)_{h\mathcal{F}_k} \simeq (X^\land k)_{h\mathcal{F}_k(n)}$$
Let $I: \text{Top}_*^G \to \text{Top}_*^G$ be the identity functor, $\mathcal{F}_k$ the family of all graphs $H \to \Sigma_k$, for $H \leq G$, $T_k$ the partition complex of $\{1, \ldots, k\}$ (with the trivial $G$-action).

**Theorem (D)**

$$D_{nG}I(X) \simeq \Omega^\infty_G \bigvee_{k=n}^{n|G|} \left( \text{Map}_*(T_k, S_G) \wedge X^\wedge k \right)_{h\mathcal{F}_k(n)}$$

**Remark**

$$\Phi^H \bigvee_{k=n}^{n|G|} \left( \text{Map}_*(T_k, S_G) \wedge X^\wedge k \right)_{h\mathcal{F}_k(n)} \simeq \bigvee_{[H \triangleleft K]} \Phi^H \left( \text{Map}_*(T_K, S_H) \wedge X^\wedge K \right)_{h\text{Aut}_K}$$

where $T_K$ is the partition complex of the $H$-set $K$. 
The End

Thank you!