Lannes' T vs Harish-Chandra restriction

(joint work with Nguyen Dang Ho Hai & Lionel Schwartz)

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Abstract

In the 1980's, Adams, Gunawardena and Miller computed Steenrod-algebra maps between elementary abelian group mod. p cohomologies. As a consequence, their decomposition is governed by the modular representations of the semi-groups of square matrices. Indeed, given $V_n = (\mathbb{Z}/p)^n$, and for a summand P in $\mathbb{F}_p[M_n(\mathbb{F}_p)]$, $L_P := \mathrm{Hom}_{M_n}(P, H^* V_n)$ is a summand in $H^* V_n$. Applying Lannes' T functor on these defines an intriguing construction for representation theorists. We show that $T(L_P) \cong L_P \oplus H^* V_1 \otimes L_{\delta(P)}$, defining a functor δ from $\mathbb{F}_p[M_n(\mathbb{F}_p)]$-projectives to $\mathbb{F}_p[M_{n-1}(\mathbb{F}_p)]$-projectives, and we relate this new functor δ to classical constructions in the representation theory of the general linear groups.

Main result

Theorem (VF, Nguyen Dang Ho Hai & Lionel Schwartz)

Let E be a reduced unstable injective module. Then there exists a (up to isomorphism) unique reduced injective unstable module $\delta(E)$ such that:

$$\mathrm{T}(E) \cong E \oplus H \otimes \delta(E) \qquad \textit{with } H := H^*(\mathbb{Z}/p, \mathbb{F}_p)$$

Computations of Harris & Shank [1] support the statement.
The case when $E = \mathrm{St}_n$ is the image of a Steinberg idempotent in $H^*(V_n, \mathbb{F}_p)$, $V_n = (\mathbb{Z}/p)^n$, appears there, and $\delta(E)$ is again a Steinberg:

$$\delta(\mathrm{St}_n) = \mathrm{St}_{n-1}$$

These authors admit to have been "somewhat surprised" by this "intriguing isomorphism".

[1] John C. Harris & R. James Shank, *Lannes' T functor on summands of $H^*(B(\mathbf{Z}/p)^s)$*, Trans. Amer. Math. Soc. **333** (1992), 579–606.

T's spectrum

Our interest was tickled by Nguyen's proof [2] [3] of Schwartz's conjecture on the action of T on reduced injectives.
Denote by K_n, the Grothendieck group of summands in $H^*(V_n, \mathbb{F}_p)$ where $V_n = (\mathbb{Z}/p)^n$. When E is in K_n, $H \otimes E$ is in K_{n+1}.

Corollary (NDHH 2015)

The operator induced by Lannes T-functor on K_n is diagonalizable over $\mathbb{Q}$, with eigenvalues $1, \ldots p^i, \ldots p^{n-1}, p^n$ and respective multiplicities $p^n - p^{n-1}, \ldots p^{n-i} - p^{n-i-1}, \ldots p - 1, 1$.

Indeed, when E is in K_n, $\delta(E)$ is in K_{n-1}.
This is what Nguyen uses in his CRAS note to deduce the corollary.
Although he uses topological arguments there.

[2] Nguyen Dang Ho Hai, *On a conjecture of Lionel Schwartz about the eigenvalues of Lannes' T-functor*, C. R. Math. Acad. Sci. Paris **353** (2015), 197–202.

[3] *A proof of Schwartz's conjecture about the eigenvalues of Lannes' T-functor*, J. Algebra **445** (2016), 115–124.

We wanted to prove that an isomorphism

$$\mathrm{T}(E) \cong E \oplus H \otimes \delta(E)$$

holds as unstable modules, not just with a virtual $\delta(E)$ in K_{n-1}.

1984

Let $\mathbb{F}_2[t_1, \ldots, t_n]$ be the ring of n-variable polynomial, seen as an unstable algebra over the Steenrod algebra, *i. e.* $H^*(V_n, \mathbb{F}_2)$ where $V_n = (\mathbb{Z}/2)^n$.
Let M_n be the semi-group of $n \times n$ matrices with $\mathbb{F}_2$ entries, acting on polynomials by linear substitutions of variables.
Naturality of Steenrod operations tells that the actions commute. Indeed:

Theorem (Adams-Gunawardena-Miller)

$\mathbb{F}_p[\mathrm{M}_n(\mathbb{F}_p)] \to \mathrm{End}_{\mathcal{U}}(H^* V_n)$ *is an isomorphism.*

The letter $\mathcal{U}$ is only here to stress the unstable condition. It is essential in the celebrated:

Theorem (Carlsson, Miller, Lannes-Zarati)

$H^* V_n$ is an injective unstable.

1980's

Given a projective $\mathbb{F}_p[\mathrm{M}_n(\mathbb{F}_p)]$-module P, put

$$L_P := \mathrm{Hom}_{\mathrm{M}_n-\mathrm{mod}}(P, H^*V_n).$$

A direct summand in an injective unstable, L_P is an injective unstable.
Also, if P is an indecomposable $\mathbb{F}_p[\mathrm{M}_n(\mathbb{F}_p)]$-projective, then L_P is an indecomposable injective unstable.
The Adams-Gunawardena-Miller theorem leads to:

Theorem (Harris-Kuhn, 1988)

The correspondence $P \mapsto L_P$ defines an equivalence from the category of projective $\mathbb{F}_p[\mathrm{M}_n(\mathbb{F}_p)]$-modules, to the category of injective unstable modules which are direct sums of indecomposable factors of H^*V_n, with inverse functor $\mathrm{Hom}_{\mathcal{U}}(-, H^*V_n)$.

Lannes' T functor

Lannes' magic functor $\mathrm{T} : \mathcal{U} \to \mathcal{U}$ is left adjoint to the functor $M \mapsto M \otimes H$ where $H := H^*\mathbb{Z}/p$.
The exactness of Lannes' T-functor reflects the $\mathcal{U}$-injectivity of H.
We use the reduced version $\overline{\mathrm{T}}$, a left adjoint to the tensor product with $\overline{H} = \overline{H}^*\mathbb{Z}/p$.

Theorem (bis)

Let E be a reduced injective unstable module. Then there exists a (up to $\mathcal{U}$-isomorphism) uniquely defined reduced injective unstable module $\delta(E)$ such that:

$$\overline{\mathrm{T}}(E) \cong H \otimes \delta(E)$$

Moreover, if E is in H^*V_n, then $\delta(E)$ is in H^*V_{n-1}.

Lannes' T-functor: examples

A classical computation by Lannes gives an $\mathrm{End}(V)$-isomorphism of unstable modules

$$\mathrm{T}H^*(V) = H^*(V)^V \cong \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p[V], H^*(V)) = \mathbb{F}_p^V \otimes H^*(V)\,.$$

The right $\mathrm{End}(V)$-action on the right-hand side is given by:

$$(\phi \otimes x).\varphi = (\phi \circ \varphi) \otimes \varphi^*(x)\,.$$

For the reduced version

$$\overline{\mathrm{T}}H^*(V) = \mathcal{J}(V) \otimes H^*V,$$

where $\mathcal{J}(V)$ denotes the quotient of the vector space of set maps, $\mathbb{F}_p^V$, by the sub-space of constant maps.
As a module over the Steenrod algebra, this is just $p^n - 1$ copies of H^*V.

$$\overline{\mathrm{T}}(L_P) \cong \mathrm{Hom}_{\mathrm{M}_n-\mathrm{mod}}(P, \mathcal{J}(V_n) \otimes H^*V_n).$$

This is a submodule in direct sums of copies of H^*V_n, and it is $\mathcal{N}il$-closed.

GL_n variant

To use classical GL_n-representation theory, consider, for a projective P:

$$M_P := \mathrm{Hom}_{\mathrm{GL}_n-\mathrm{mod}}(P, H^*V_n).$$

The best example is for $P = \mathrm{St}_n$, the Steinberg module.
For $p = 2$, $M_{\mathrm{St}_n} \cong L_{\mathrm{St}_n} \oplus L_{\mathrm{St}_{n-1}}$ – a typical decomposition.
Let us illustrate in proving Harris & Shank's result.
We want to use adjonctions, in particular restriction and induction.

$$\begin{aligned} \overline{\mathrm{T}}(M_{\mathrm{St}_n}) &\cong \mathrm{Hom}_{\mathrm{GL}_n}(\mathrm{St}_n, \mathcal{J}(V_n) \otimes H^*V_n) \\ &\cong \mathrm{Hom}_{\mathrm{GL}_n}(\mathcal{J}(V_n)^\# \otimes \mathrm{St}_n, H^*V_n) \end{aligned}$$

A character computation identifies the left-hand side:

Lemma (NDHH)

$\mathcal{J}(V_n)^\# \otimes \mathrm{St}_n$ is isomorphic to $\mathrm{Ind}_{\mathrm{GL}_{n-1}}^{\mathrm{GL}_n}(\mathrm{St}_{n-1})$.

The Steinberg case

$$\begin{aligned}
\overline{\mathrm{T}}(M_{\mathrm{St}_n}) &\cong \mathrm{Hom}_{\mathrm{GL}_n}(\mathrm{St}_n, \mathcal{J}(V_n) \otimes H^* V_n) \\
&\cong \mathrm{Hom}_{\mathrm{GL}_n}(\mathcal{J}(V_n)^{\#} \otimes \mathrm{St}_n, H^* V_n) \\
&\cong \mathrm{Hom}_{\mathrm{GL}_n}\big(\mathsf{Ind}_{\mathrm{GL}_{n-1}}^{\mathrm{GL}_n}(\mathrm{St}_{n-1}), H^* V_n\big) \\
&\cong \mathrm{Hom}_{\mathrm{GL}_{n-1}}\big(\mathrm{St}_{n-1}, \mathsf{Res}_{\mathrm{GL}_{n-1}}^{\mathrm{GL}_n}(H^* V_n)\big) \\
&\cong \mathrm{Hom}_{\mathrm{GL}_{n-1}}(\mathrm{St}_{n-1}, H^* V_{n-1} \otimes H) \\
&\cong \mathrm{Hom}_{\mathrm{GL}_{n-1}}(\mathrm{St}_{n-1}, H^* V_{n-1}) \otimes H \\
&\cong \mathrm{Hom}_{\mathrm{GL}_{n-1}}(\mathrm{St}_{n-1}, H^* V_{n-1}) \otimes H \\
&\cong M_{\mathrm{St}_{n-1}} \otimes H.
\end{aligned}$$

The Steinberg case extended

$$\begin{aligned}
\overline{\mathrm{T}}(M_{\mathrm{St}_n \otimes X}) &\cong \mathrm{Hom}_{\mathrm{GL}_n}(\mathrm{St}_n \otimes X, \mathcal{J}(V_n) \otimes H^* V_n) \\
&\cong \mathrm{Hom}_{\mathrm{GL}_n}(\mathcal{J}(V_n)^{\#} \otimes \mathrm{St}_n, X^{\#} \otimes H^* V_n) \\
&\cong \mathrm{Hom}_{\mathrm{GL}_n}\big(\mathsf{Ind}_{\mathrm{GL}_{n-1}}^{\mathrm{GL}_n}(\mathrm{St}_{n-1}), X^{\#} \otimes H^* V_n\big) \\
&\cong \mathrm{Hom}_{\mathrm{GL}_{n-1}}\big(\mathrm{St}_{n-1}, \mathsf{Res}_{\mathrm{GL}_{n-1}}^{\mathrm{GL}_n}(X^{\#} \otimes H^* V_n)\big) \\
&\cong \mathrm{Hom}_{\mathrm{GL}_{n-1}}\big(\mathrm{St}_{n-1}, \mathsf{Res}_{\mathrm{GL}_{n-1}}^{\mathrm{GL}_n}(X^{\#}) \otimes H^* V_{n-1} \otimes H\big) \\
&\cong \mathrm{Hom}_{\mathrm{GL}_{n-1}}\big(\mathrm{St}_{n-1}, \mathsf{Res}_{\mathrm{GL}_{n-1}}^{\mathrm{GL}_n}(X^{\#}) \otimes H^* V_{n-1}\big) \otimes H \\
&\cong \mathrm{Hom}_{\mathrm{GL}_{n-1}}\big(\mathrm{St}_{n-1} \otimes \mathsf{Res}_{\mathrm{GL}_{n-1}}^{\mathrm{GL}_n} X, H^* V_{n-1}\big) \otimes H \\
&\cong M_{\mathrm{St}_{n-1} \otimes \mathsf{Res}_{\mathrm{GL}_{n-1}}^{\mathrm{GL}_n} X} \otimes H.
\end{aligned}$$

A theorem of George Lusztig from 1976 (also due to John W. Ballard) states that a projective GL_n-module can always be written as $\mathrm{St}_n \otimes X$ where X is a virtual GL_n-module. This proves that, for P a projective GL_n-module,

$$\overline{\mathrm{T}}(M_P) \cong H \otimes M'$$

where M' is a *formal* sum of direct summands of $H^* V_{n-1}$.

Now let's get real, first for GL_n.

The GL_n case

Instead of $\mathcal{J}(V)$, we may consider its Kuhn dual
$\mathcal{I}(V) := \mathcal{J}(V^*)^* =$ the augmentation ideal in the group algebra of V^*.

This doesn't change much:
because $\mathcal{I}(V)$ and $\mathcal{J}(V)$ have the same Jordan-Hölder subquotients, the GL_n-projectives $P \otimes \mathcal{J}(V_n)^{\#}$ and $P \otimes \mathcal{I}(V_n)^{\#}$ are isomorphic.

$$\begin{aligned}\overline{\mathrm{T}}(M_P) &= \mathrm{Hom}_{\mathrm{GL}_n}(P, \mathcal{J}(V_n) \otimes H^* V_n) \cong \mathrm{Hom}_{\mathrm{GL}_n}(P \otimes \mathcal{J}(V_n)^{\#}, H^* V_n) \\ &\cong \mathrm{Hom}_{\mathrm{GL}_n}(P \otimes \mathcal{I}(V_n)^{\#}, H^* V_n) \cong \mathrm{Hom}_{\mathrm{GL}_n}(P, \mathcal{I}(V_n) \otimes H^* V_n).\end{aligned}$$

The point is that the right-hand side is an H unstable module: an unstable module provided with an H-module structure, for which the Cartan formula holds. Explicitely, one gets a M_n-equivariant H-action:

$$u.((\mu) \otimes x) = (\mu) \otimes \mu^*(u)x$$

for μ in V_n^*, $(\mu) = [\mu] - [0]$ in $\mathcal{I}(V_n)$, u in H, and x in $H^*(V_n)$.

H-module structure

The H-action on $\mathcal{I}(V_n) \otimes H^*V_n$ is free, and
the quotient $\mathbb{F}_p \otimes_H (\mathcal{I}(V) \otimes H^*V)$ is a reduced injective unstable module.
Explicitely, there is an $\mathrm{End}(V)$-equivariant iso. of unstable modules:

$$\mathbb{F}_p \otimes_H (\mathcal{I}(V) \otimes H^*V) \to \bigoplus_{\mu \in V^* \setminus \{0\}} H^*(\mathrm{Ker}\,\mu)$$

sending $1 \otimes ((\mu) \otimes x)$ to $\mu^*(x)$ in the summand $H^*(\mathrm{Ker}\,\mu)$.

To sum up, we know that $\overline{\mathrm{T}}(M_P) \cong \mathrm{Hom}_{\mathrm{GL}_n}(P, \mathcal{I}(V_n) \otimes H^*V_n)$ is an unstable free H-module with reduced injective quotient. We want to factor it with H. This is a case to be found in Bourguiba's 2009 paper:

Bourguiba's criterium

To sum up: $\overline{\mathrm{T}}(M_P) \cong \mathrm{Hom}_{\mathrm{GL}_n}(P, \mathcal{I}(V_n) \otimes H^*V_n)$ is an H-free unstable module with reduced injective quotient. We want to factor H.
This is a case to be found in Bourguiba's 2009 paper[4]:

Theorem (Bourguiba 2009, Theorem 3.2.1)

Let E be an H-free unstable module and let $\mathcal{E}(E)$ be its injective hull (as an H-unstable module). We suppose that $\mathbb{F}_p \otimes_H E$ is reduced and let I be its injective hull (as an unstable module).
Then $\mathcal{E}(E)$ is isomorphic, as an H unstable module, to $H \otimes I$.

We conclude: $\overline{\mathrm{T}}(M_P) \cong H \otimes \mathrm{Hom}_{\mathrm{GL}_n}(P, \bigoplus_{\mu \in V_n^* \setminus \{0\}} H^*(\mathrm{Ker}\,\mu))$

[4] Dorra Bourguiba, *On the classification of unstable H^*V-A-modules*, J. Homotopy Relat. Struct. **4** (2009), 69–82.

The M_n case

When dealing with all matrices, we cannot use the contragredient, so we only get the isomorphism after taking $\mathrm{Hom}_{\mathrm{M}_n}$:

$$\overline{\mathrm{T}}(L_P) \cong \mathrm{Hom}_{\mathrm{M}_n}(P, \mathcal{I}(V_n) \otimes H^* V_n) \cong H \otimes \mathrm{Hom}_{\mathrm{M}_n}(P, \bigoplus_{\mu \in V_n^* \setminus \{0\}} H^*(\mathrm{Ker}\, \mu))$$

Theorem

The functor $\mathrm{Hom}_{\mathrm{M}_n}(-, \bigoplus_{\mu \in V_n^* \setminus \{0\}} H^*(\mathrm{Ker}\, \mu))$ induces an exact functor from $M_n - \mathrm{proj}$ to $M_{n-1} - \mathrm{proj}$, which we denote again by δ_n, such that for each P in $\mathrm{M}_n - \mathrm{proj}$, there is an isomorphism of unstable modules:

$$\overline{\mathrm{T}}(L_P) \cong H \otimes L_{\delta_n(P)}.$$

The isomorphism is not natural, but the functor δ_n is quite explicit.

Interpretation

So far, we implicitely considered V_{n-1} to be the hyperplane of zero last coordinate in V_n; and the group GL_{n-1} as a subgroup of GL_n *via* the inclusion

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

The (parabolic) subgroup of GL_n which stabilizes V_{n-1} is a semi-direct product LU of two subgroups:
the Levi subgroup $L := \mathrm{GL}_{n-1} \times \mathrm{GL}_1$, and a unipotent subgroup U.
Inflation is the simplest way to extend a representation from a subgroup H to HU, by just letting the elements of the normal subgroup U act by the identity. It is right adjoint to taking U-coinvariant.

Start with the inclusion

$$H^* V_{n-1} \hookrightarrow \bigoplus_{\mu \neq 0} H^* \operatorname{Ker} \mu$$

of $H^* V_{n-1}$ as the factor indexed by the n-th coordinate form μ_0.
Every element of U fixes μ_0, and is the identity on V_{n-1}.
Thus, the inclusion inflates to a $\mathrm{GL}_{n-1} U$-equivariant map:

$$\operatorname{Inf}_{\mathrm{GL}_{n-1}}^{\mathrm{GL}_{n-1} U} H^* V_{n-1} \hookrightarrow \bigoplus_{\mu \neq 0} H^* \operatorname{Ker} \mu .$$

It corresponds by Ind/Res adjunction to the GL_n-equivariant map

$$\mathbb{F}_p[\mathrm{GL}_n] \bigotimes_{\mathbb{F}_p[\mathrm{GL}_{n-1} U]} H^* V_{n-1} \cong \operatorname{Ind}_{\mathrm{GL}_{n-1} U}^{\mathrm{GL}_n} \operatorname{Inf}_{\mathrm{GL}_{n-1}}^{\mathrm{GL}_{n-1} U} H^* V_{n-1} \to \bigoplus_{\mu \neq 0} H^* \operatorname{Ker} \mu ,$$

sending $g \bigotimes x$ to $g^*(x)$ in $H^* \operatorname{Ker}(\mu_0 \circ g)$.
It is surjective because GL_n acts transitively on $\{\operatorname{Ker} \mu\}$.
Since the vector space on the left-hand side is isomorphic to $|GL_n| / |\mathrm{GL}_{n-1} U| = p^n - 1$ copies of $H^* V_{n-1}$, the map is bijective.

Comparison with Harish-Chandra restriction

$$\begin{aligned} \bigoplus_{\mu \in V_n^*} H^* \operatorname{Ker} \mu &\cong \operatorname{Ind}_{\mathrm{GL}_{n-1} U}^{\mathrm{GL}_n} \operatorname{Inf}_{\mathrm{GL}_{n-1}}^{\mathrm{GL}_{n-1} U} H^* V_{n-1} \\ &\cong \operatorname{Ind}_{LU}^{\mathrm{GL}_n} \operatorname{Ind}_{\mathrm{GL}_{n-1} U}^{LU} \operatorname{Inf}_{\mathrm{GL}_{n-1}}^{\mathrm{GL}_{n-1} U} H^* V_{n-1} \\ &\cong \operatorname{Ind}_{LU}^{\mathrm{GL}_n} \operatorname{Inf}_{L}^{LU} \operatorname{Ind}_{\mathrm{GL}_{n-1}}^{L} H^* V_{n-1} \end{aligned}$$

The composite $\operatorname{Ind}_{LU}^{\mathrm{GL}_n} \operatorname{Inf}_{L}^{LU}$ is known as Harish-Chandra induction: starting with a L-module, inflate the action to the parabolic LU by letting U act by the identity; then apply induction to GL_n.
Harish-Chandra restriction R_L is defined as the (left) adjoint of Harish-Chandra induction. Explicitly, starting with a $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]$-module X, the Harish-Chandra restriction of X is obtained by restricting to the parabolic subgroup, followed by taking coinvariants of the corresponding unipotent U:

$$\mathrm{R}_L X = (\operatorname{Res}_{LU}^{GL_n} X)_U .$$

Comparison with Harish-Chandra restriction

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_n}(P, \bigoplus H^* \operatorname{Ker}\mu) &\cong \mathrm{Hom}_{\mathrm{GL}_n}(P, \operatorname{Ind}_{LU}^{\mathrm{GL}_n} \operatorname{Inf}_L^{LU} \operatorname{Ind}_{\mathrm{GL}_{n-1}}^L H^* V_{n-1}) \\ &\cong \mathrm{Hom}_L(\mathrm{R}_L\, P, \operatorname{Ind}_{\mathrm{GL}_{n-1}}^L H^* V_{n-1}) \\ &\cong \mathrm{Hom}_{\mathrm{GL}_{n-1}}(\operatorname{Res}_{\mathrm{GL}_{n-1}}^L \mathrm{R}_L\, P, H^* V_{n-1})\end{aligned}$$

Theorem

For each P in $\mathrm{GL}_n{-}\mathrm{proj}$, there is an isomorphism of unstable modules:

$$\overline{\mathrm{T}}(M_P) \cong H \otimes M_{\delta_n(P)},$$

where $\delta_n(P)$ in $\mathrm{GL}_{n-1}{-}\mathrm{proj}$ is the Harish-Chandra restriction for the Levi subgroup $\mathrm{GL}_{n-1} \times \mathrm{GL}_1$, restricted to GL_{n-1}.