# The Poincaré–Hopf theorem for line fields (revisited)

(joint with D. Crowley)

Mark Grant

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## Line fields

Let $M^m$ be a smooth manifold of dimension $m \geq 2$.

**Definition**

A line field on $M$ is a smooth section $\xi : M \to PTM$ of the projectivized tangent bundle.

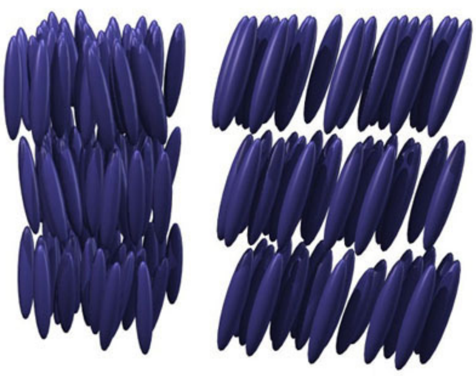
In other words, a line field is a smooth assignment

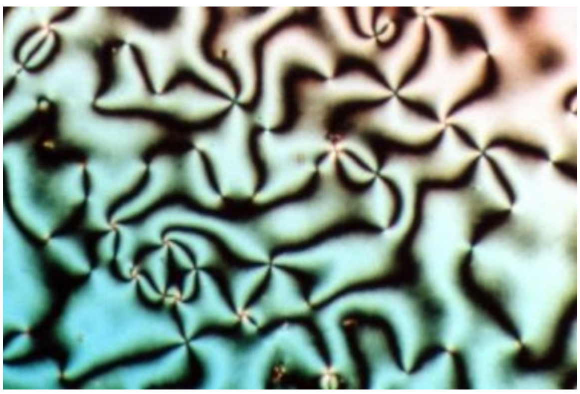
$$x \mapsto \xi(x) \subset TM_x$$

of a one-dimensional subspace of the tangent space at each point.



Line fields, or nematic fields, are of interest in soft-matter physics, where they are used to model nematic liquid crystals.





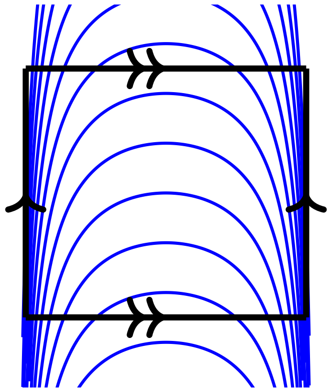
(Images: `https://en.wikipedia.org/wiki/Liquid_crystal`)

A nowhere zero vector field $v : M \to TM$ gives rise to a line field by setting

$$\xi(x) = \langle v(x) \rangle \subset TM_x$$

to be the line spanned by $v(x)$.

However, not every line field can be lifted to a nowhere zero vector field.





### Proposition

A closed manifold $M$ admits a line field if and only if it admits a nowhere zero vector field.

Proof: A line field $\xi$ on $M$ may be viewed as a line sub-bundle $\xi \subset TM$.

Fix a metric on $M$, then the sphere bundle

$$p_\xi : \widetilde{M} := S(\xi) \to M$$

is the associated double cover.

Note that $\widetilde{M}$ has a canonical nowhere zero vector field which lifts $p_\xi^*\xi$.

By the multiplicativity of the Euler characteristic for covers,

$$0 = \chi(\widetilde{M}) = 2\,\chi(M),$$

hence $\chi(M) = 0$ and $M$ admits a nowhere zero vector field. $\square$



**Theorem (Poincaré–Hopf)**

Let $v : M \to TM$ be a vector field with isolated zeroes at $x_1, \ldots, x_n \in M$. Then

$$\sum_{i=1}^{n} \mathrm{ind}_v(x_i) = \chi(M).$$

The index $\mathrm{ind}_v(x_i) \in \mathbb{Z}$ is the degree of the composition

$$f : S \xrightarrow{v|_S} STM|_S \xrightarrow{\Phi} S \times S^{m-1} \xrightarrow{\pi_2} S^{m-1},$$

where:

- $v|_S$ is the restriction of (the normalization of) $v$ to a small sphere $S$ centred at $x_i$;
- $\Phi$ is a trivialisation, and
- $\pi_2$ is projection onto the second factor.

## Poincaré–Hopf Theorem for line fields

### Definition

A line field on $M$ with singularities at $x_1, \dots, x_n \in M$ is a line field on the complement $M \setminus \{x_1, \dots, x_n\}$.

A vector field with zeroes determines a line field with singularities, but a line field with singularities need not lift to a vector field.

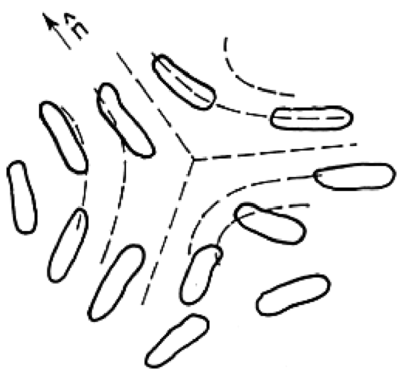
### Question

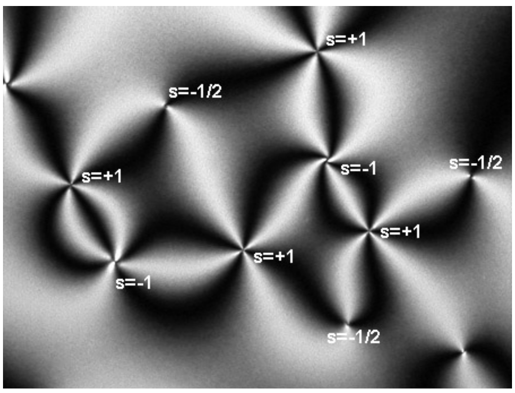
What is the analogue of Poincaré–Hopf for line fields with singularities?



The singularities are known as topological defects in the Physics literature.

Of particular interest are point defects in $2$ and $3$ dimensions, and line defects or disclinations in $3$ dimensions (which may be knotted).






(Images: `http://www.lassp.cornell.edu/sethna/OrderParameters/TopologicalDefects.html`, `http://www.personal.kent.edu/~bisenyuk/liquidcrystals/textures1.html`)

# Hopf's result

**Theorem (Hopf)**

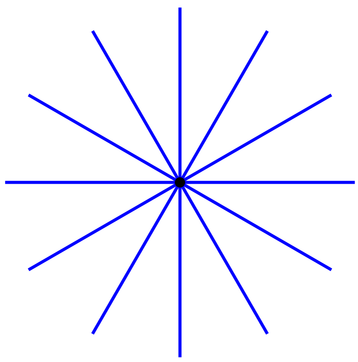
A line field $\xi$ with singularities $x_1, \dots, x_n$ on a closed orientable surface $\Sigma$ has

$$\sum_{i=1}^{n} \mathrm{h\,ind}_\xi(x_i) = \chi(\Sigma).$$

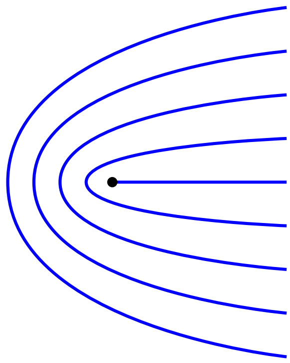
The Hopf index $\mathrm{h\,ind}_\xi(x_i) \in \frac{1}{2}\mathbb{Z}$ is the number of total rotations made by $\xi$ as a simple closed curve around $x_i$ is traversed.

Reference: H. Hopf, *Differential Geometry in the Large*, LNM 1000, (1983) (Based on lectures given at Stanford University in 1956).

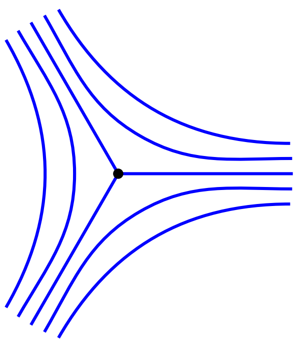




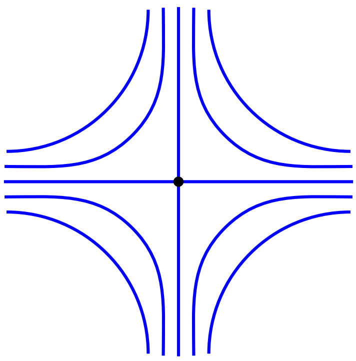
(a) $\mathrm{h\,ind}_\xi(x) = 1$



(b) $\mathrm{h\,ind}_\xi(x) = \frac{1}{2}$



(c) $\mathrm{h\,ind}_\xi(x) = -\frac{1}{2}$



(d) $\mathrm{h\,ind}_\xi(x) = -1$

Line field singularities and their Hopf indices.

# Markus' result

## Definition

A singularity $x_i$ of a line field $\xi$ on $M^m$ is called (non)-orientable if the restriction of $\xi$ to a small sphere $S$ centred at $x_i$ lifts (does not lift) to a vector field.

Equivalently, $x_i$ is (non)-orientable if the restriction to $S$ of the associated double cover $p_\xi|_S : \widetilde{S} \to S$ is (non)-trivial.

If $m = 2$, then $x_i$ is orientable if and only if $\mathrm{h\,ind}_\xi(x_i) \in \mathbb{Z}$.

If $m > 2$, then all singularities are orientable.



The Markus index $\mathrm{m\,ind}_\xi(x_i) \in \mathbb{Z}$ is defined as follows:

For $m$ even, it is the degree of the composition

$$f : S \xrightarrow{\xi|_S} PTM|_S \xrightarrow{\Phi} S \times \mathbb{R}P^{m-1} \xrightarrow{\pi_2} \mathbb{R}P^{m-1}.$$

For $m \geq 3$ odd, orienting $\xi$ near $x_i$ gives a lift $\tilde{f} : S \to S^{m-1}$ of $f : S \to \mathbb{R}P^{m-1}$. Choose base points and suspend, and take the degree of the composition

$$S^m \xrightarrow{\Sigma\tilde{f}} S^m \longrightarrow \mathbb{R}P^m.$$

**Theorem (Markus)**

A line field $\xi$ with singularities $x_1, \ldots, x_n$ on a closed manifold $M^m$ has

$$\sum_{i=1}^{n} \mathrm{m\,ind}_\xi(x_i) = 2\chi(M) - k,$$

where $k$ is the number of non-orientable singularities.

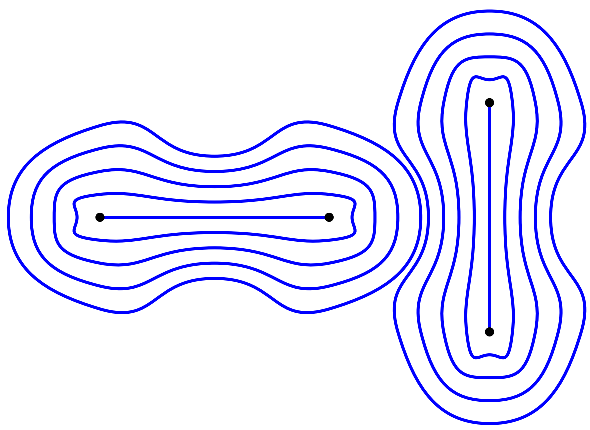
Reference: L. Markus, *Line element fields and Lorentz structures on differentiable manifolds*, Ann. Math. 62, (1955)

Unfortunately, there are counter-examples to Markus' Theorem for $m = 2$ and $m \geq 3$ odd.



## Example: The baseball

There is a line field on $S^2$, known colloquially as "the baseball", with four non-orientable singularities of Hopf index $\frac{1}{2}$ and Markus index $1$.

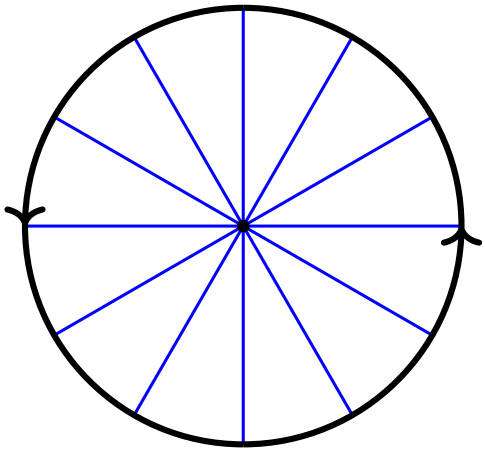


This contradicts Markus' Theorem, since

$$\sum_{i=1}^{n} \mathrm{m\,ind}_\xi(x_i) = 4 \neq 0 = 2\chi(S^2) - 4.$$

## Example: The hedgehog

This is a line field on $\mathbb{R}P^m$ with a single orientable singularity of Hopf index $1$ and Markus index $2$.



For $m \geq 3$ odd this contradicts Markus' Theorem, since

$$\sum_{i=1}^{n} \operatorname{m\,ind}_\xi(x_i) = 2 \neq 0 = 2\chi(\mathbb{R}P^m).$$



## Our result

We define the projective index by

$$\operatorname{p\,ind}_\xi(x_i) = \begin{cases} \deg(f) & \in \mathbb{Z} \quad \text{if } m \text{ even,} \\ \deg_2(f) & \in \mathbb{Z}/2 \quad \text{if } m \text{ odd,} \end{cases}$$

where $f : S^{m-1} \to \mathbb{R}P^{m-1}$ is the composition

$$f : S \xrightarrow{\xi|_S} PTM|_S \xrightarrow{\Phi} S \times \mathbb{R}P^{m-1} \xrightarrow{\pi_2} \mathbb{R}P^{m-1}.$$

# Our result

## Theorem (Crowley–G.)

A line field $\xi$ with singularities $x_1, \ldots, x_n$ on a closed manifold $M^m$ has

$$\sum_{i=1}^{n} \mathrm{p}\,\mathrm{ind}_\xi(x_i) = 2\chi(M).$$

The equality is congruence mod $2$ when $m$ is odd.



## Remarks

This corrects Markus' Theorem, and extends Hopf's Theorem to dimensions $m > 2$.

Our proof is similar to that of Markus, but we introduce normal indices to clarify some issues when $m = 2$.

# Normal indices

Let $x$ be an isolated zero of the vector field $v : M \to TM$. Recall that $\mathrm{ind}_v(x)$ is the degree of the composition

$$f : S \xrightarrow{v|_S} STM|_S \xrightarrow{\Phi} S \times S^{m-1} \xrightarrow{\pi_2} S^{m-1}.$$

If $a \in S^{m-1}$ is a regular value of $f$, then $v|_S$ is transverse to the embedding $\sigma = \sigma_a : S \hookrightarrow STM|_S$ given by

$$\sigma(y) = \Phi^{-1}(y, a).$$

Then $\mathrm{ind}_v(x)$ equals the oriented intersection number

$$\sigma(S) \pitchfork v(S) \in \mathbb{Z}.$$



Suppose $M$ endowed with a Riemannian metric. Then the outward unit normal to $S$ defines an embedding $\eta : S \hookrightarrow STM|_S$.

### Definition

The normal index $\mathrm{ind}_v^\perp(x) \in \mathbb{Z}$ is defined to be the oriented intersection number

$$\eta(S) \pitchfork v(S) \in \mathbb{Z}.$$

The normal index counts the number of times $v$ points outwards on $S$ (with signs).

**Lemma**

We have
$$\operatorname{ind}_v^{\perp}(x) = \operatorname{ind}_v(x) + (-1)^{m-1}.$$

Proof: Calculate intersection numbers in
$$H^*(S \times S^{m-1}) \cong H^*(S) \otimes H^*(S^{m-1}).$$

The Poincaré dual of $\Phi_*\sigma_*([S])$ is $(-1)^{m-1} \times \beta$, and the Poincaré dual of $\Phi_*\eta_*([S])$ is $\alpha \times 1 + (-1)^{m-1} \times \beta$.

Take cup products with the Poincaré dual of $\Phi_* v_*([S])$ and compare to give the result. □



Now let $x$ be an isolated singularity of the line field $\xi : M \to PTM$. Recall that $\operatorname{p\,ind}_\xi(x)$ is the degree of the composition
$$f : S \xrightarrow{\xi|_S} PTM|_S \xrightarrow{\Phi} S \times \mathbb{R}P^{m-1} \xrightarrow{\pi_2} \mathbb{R}P^{m-1}.$$

If $a \in \mathbb{R}P^{m-1}$ is a regular value of $f$, then $\xi|_S$ is transverse to the embedding $\sigma = \sigma_a : S \hookrightarrow PTM|_S$ given by
$$\sigma(y) = \Phi^{-1}(y, a).$$

Then $\operatorname{p\,ind}_\xi(x)$ equals the intersection number
$$\operatorname{p\,ind}_\xi(x) = \begin{cases} \sigma(S) \pitchfork \xi(S) & \in \mathbb{Z} \quad \text{if } m \text{ even,} \\ \sigma(S) \pitchfork_2 \xi(S) & \in \mathbb{Z}/2 \quad \text{if } m \text{ odd.} \end{cases}$$

The normal line to $S$ defines an embedding $\eta : S \hookrightarrow PTM|_S$.

**Definition**

The normal projective index is defined by

$$\mathrm{p\,ind}_\xi^\perp(x) = \begin{cases} \eta(S) \pitchfork \xi(S) & \in \mathbb{Z} \quad \text{if } m \text{ even,} \\ \eta(S) \pitchfork_2 \xi(S) & \in \mathbb{Z}/2 \quad \text{if } m \text{ odd.} \end{cases}$$

The normal projective index counts the number of times $\xi$ is normal to $S$ (with signs if $m$ is even).



**Lemma**

When $m$ is even, we have

$$\mathrm{p\,ind}_\xi^\perp(x) = \mathrm{p\,ind}_\xi(x) - 2.$$

Proof: This is a calculation in the integral (co)homology of $S \times \mathbb{R}P^{m-1}$, analogous to the previous Lemma. $\square$

**Lemma**

When $m \geq 3$ is odd, we have

$$\operatorname{p\,ind}_\xi(x) \equiv \operatorname{p\,ind}_\xi^\perp(x) \equiv 0 \in \mathbb{Z}/2.$$

Proof: The map $f : S \to \mathbb{R}P^{m-1}$ lifts through the standard double cover $S^{m-1} \to \mathbb{R}P^{m-1}$, and therefore $\operatorname{p\,ind}_\xi(x) = \deg_2(f) \equiv_2 0$. Since $\sigma$ and $\eta$ represent the same mod 2 homology class, the result follows. $\square$



# The proof

**Theorem (Crowley–G.)**

A line field $\xi$ with singularities $x_1, \ldots, x_n$ on a closed manifold $M^m$ has

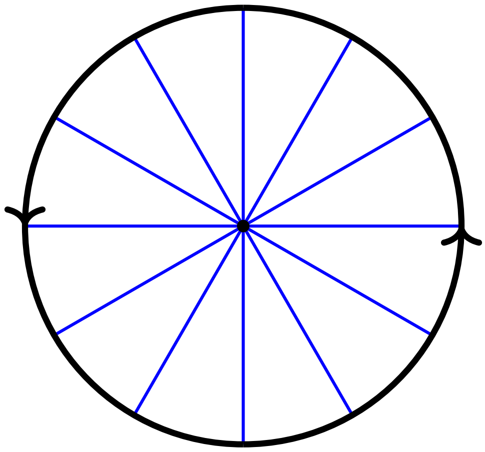
$$\sum_{i=1}^{n} \operatorname{p\,ind}_\xi(x_i) = 2\chi(M).$$

The equality is congruence mod 2 when $m$ is odd.

Proof: When $m \geq 3$ is odd, trivial consequence of $\operatorname{p\,ind}_\xi(x_i) \equiv_2 0$.

Remark: The Markus index $\operatorname{m\,ind}_\xi(x_i) \in \mathbb{Z}$ is not well-defined for $m$ odd, since the two lifts $\widetilde{f} : S \to S^{m-1}$ differ by a map of degree $(-1)^m = -1$.

One may define an index in $\mathbb{N}_0$, but the hedgehog example suggests the above result is the best we can hope for.





So suppose $m$ even, and let $\xi$ be a line field on $M^m$ with singularities $x_1, \ldots, x_n$.

Let $D_i$ be a coordinate disk containing $x_i$ and no other singularities, and let $S_i = \partial D_i$. Then $N := M \setminus \bigcup \operatorname{int}(D_i)$ is a compact with boundary

$$\partial N \approx \bigsqcup_{i=1}^{n} S_i \approx \bigsqcup_{i=1}^{n} S^{m-1}.$$

The restriction $\xi|_N$ is a line field with associated double cover $p : \widetilde{N} \to N$.

Each restriction $p|_{S_i} : \widetilde{S}_i \to S_i$ is a double cover of $S^{m-1}$, which is trivial if and only if $x_i$ is orientable.

By gluing in $m$-disks along the boundary components of $\widetilde{N}$, we obtain a closed manifold $\widetilde{M}$ and a double cover

$$\pi : \widetilde{M} \to M$$

extending $p : \widetilde{N} \to N$.

This double cover may be branched if $m = 2$, with branch points of index 2 above the non-orientable singularities.

The line field $\xi|_N$ lifts canonically to a vector field $\widetilde{\xi}$ on $\widetilde{N}$, which extends to a vector field $v$ on $\widetilde{M}$.

Each pre-image $\pi^{-1}(x_i)$ consists of one or two isolated zeroes of $v$.

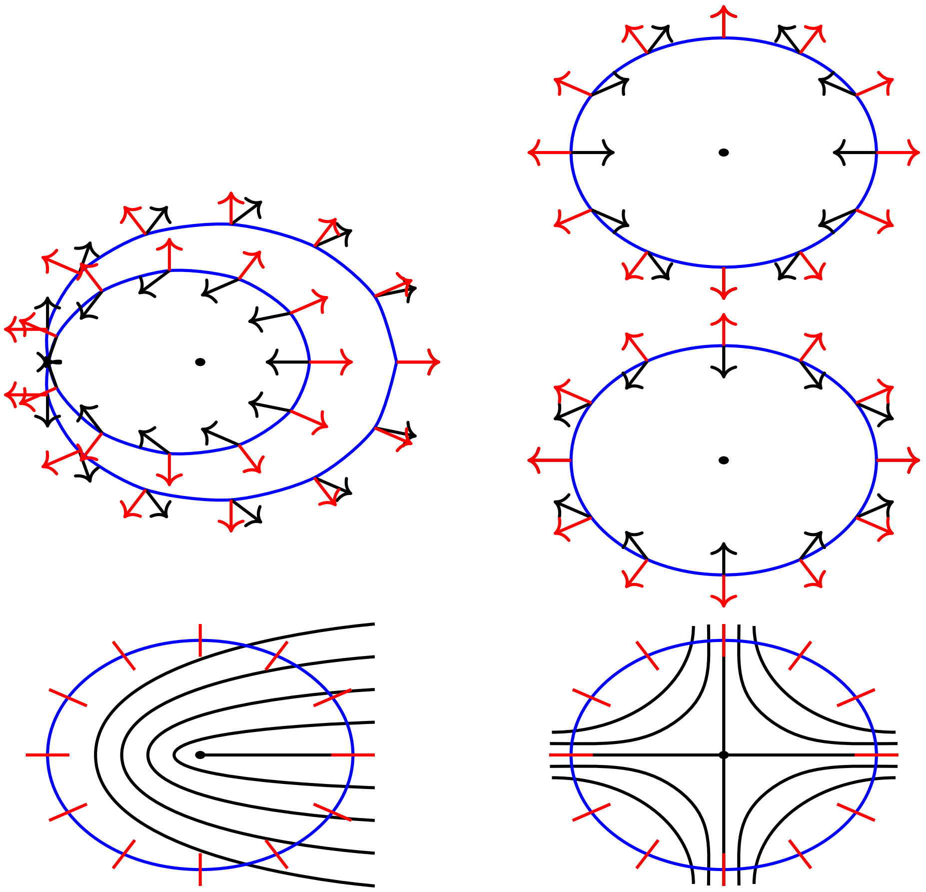


**Lemma**

For each singularity $x_i$ of $\xi$, we have

$$\mathrm{p\,ind}_{\xi}^{\perp}(x_i) = \sum_{y \in \pi^{-1}(x_i)} \mathrm{ind}_{v}^{\perp}(y).$$

This is intuitively clear: the number of times $\xi$ is normal to $S$ equals the number of times $v$ agrees with the outward normal on $\widetilde{S}$.





Proof of Lemma: The double cover $\pi : \widetilde{M} \to M$ induces a $4$-fold cover $\overline{\pi} : ST\widetilde{M}|_{\widetilde{S}} \to PTM|_S$, and there is pullback square

$$\begin{array}{ccc} \widetilde{S} \sqcup \widetilde{S} & \xrightarrow{\widetilde{\eta} \sqcup -\widetilde{\eta}} & ST\widetilde{M}|_{\widetilde{S}} \\ {\scriptstyle p \sqcup p}\downarrow & & \downarrow{\scriptstyle \overline{\pi}} \\ S & \xrightarrow{\eta} & PTM|_S \end{array}$$

where $\widetilde{\eta} : \widetilde{S} \to ST\widetilde{M}|_{\widetilde{S}}$ denotes the outward unit normal to $\widetilde{S}$.

It follows that $\overline{\pi}^* \eta_!(1) = 2\,\widetilde{\eta}_!(1)$.

By a similar argument, $\overline{\pi}^* \xi_!(1) = 2\, v_!(1)$. Therefore,

$$\begin{aligned} 4\,\mathrm{p\,ind}_\xi^\perp(x) &= 4\,\langle \eta_!(1) \cup \xi_!(1), [PTM|_S] \rangle \\ &= \langle \eta_!(1) \cup \xi_!(1), 4[PTM|_S] \rangle \\ &= \langle \eta_!(1) \cup \xi_!(1), \overline{\pi}_*[ST\widetilde{M}|_{\widetilde{S}}] \rangle \\ &= \langle \overline{\pi}^*\eta_!(1) \cup \overline{\pi}^*\xi_!(1), [ST\widetilde{M}|_{\widetilde{S}}] \rangle \\ &= \langle 2\widetilde{\eta}_!(1) \cup 2v_!(1), [ST\widetilde{M}|_{\widetilde{S}}] \rangle \\ &= 4 \sum_{y \in \pi^{-1}(x)} \mathrm{ind}_v^\perp(y), \end{aligned}$$

and the conclusion follows. $\square$



We now apply Riemann–Hurwitz and the classical Poincaré–Hopf formula.

$$\begin{aligned} 2\chi(M) = k + \chi(\widetilde{M}) &= k + \sum_{i=1}^{n} \sum_{y \in \pi^{-1}(x_i)} \mathrm{ind}_v(y) \\ &= k + \sum_{i=1}^{n} \sum_{y \in \pi^{-1}(x_i)} \Big( \mathrm{ind}_v^\perp(y) + 1 \Big) \\ &= k + (2n - k) + \sum_{i=1}^{n} \sum_{y \in \pi^{-1}(x_i)} \mathrm{ind}_v^\perp(y) \\ &= 2n + \sum_{i=1}^{n} \mathrm{p\,ind}_\xi^\perp(x_i) \\ &= 2n + \sum_{i=1}^{n} \big( \mathrm{p\,ind}_\xi(x_i) - 2 \big) \\ &= \sum_{i=1}^{n} \mathrm{p\,ind}_\xi(x_i). \qquad \square \end{aligned}$$

# Further problems

- Extend to manifolds with boundary.
- Give a differential-geometric proof in all dimensions using the higher-dimensional Gauss–Bonnet Theorem (Allendoerfer–Weil, Chern).
- What can we say about disclinations (lines of singularities)? Does the topology of $M^3$ restrict which knots and links can occur?
- Are there such Poincaré–Hopf Theorems for other types of field (such as biaxial nematics)?