Rational equivariant cohomology theories and affine formal covers of the sphere

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Models

Dream: Algebraic models for rational equivariant cohomology theories

- G-equivariant cohomology theories $=$ Ho (G-spectra)
- $\otimes\mathbb{Q}$
- $\mathcal{A}(G)$

Conjecture

- There is a Quillen equivalence $(\mathrm{G}-\mathit{spectra})/\mathbb{Q} \simeq d\mathcal{A}(G)$
- $\mathcal{A}(G)$ is small and calculable
- of injective dimension $=$ rank (G)
- a category of sheaves of modules over the space of subgroups of G

Status

Group	Isotropy	ASS	Ho	Quillen	Monoidal
Finite		GM92	(ASS).......	ScS03	B09, K14
$SO(2)$		G99	(ASS).......	GS11	BGKS16
$O(2)$		G98	(ASS).......	B12	
$SO(3)$		G01	(ASS).......	K14	
T^r		G08	...... (Quillen)	GS14	
G	free			GS14	
G	toral	G15		BGK(?)	

B=D.Barnes, K=M.Kedziorek, M=J.P.May, Sc=S.Schwede, S=B.Shipley

The proof

Theorem

For the above groups G, there is a (small, convenient, calculable) algebraic model:

$$G\text{-spectra}/\mathbb{Q} \simeq d\mathcal{A}(G)$$

Proof.

$$\begin{aligned} G\text{-spectra}/\mathbb{Q} &\overset{*}{\simeq} \text{Diagrams of module } G\text{-spectra over ring } G\text{-spectra} \\ &\simeq \text{Diagrams of module spectra over ring spectra} \\ &\simeq \text{Diagrams of modules over DGAs} \\ &\simeq \text{Diagrams of modules over rings} \simeq \mathcal{A}(G) \end{aligned}$$

□

Abelian groups

We all know about abelian groups......
Let's make them more complicated......

The ring

- Abelian groups = $\mathbb{Z}$-modules
- $$\mathbb{Z}^{\square} = \left(\begin{array}{ccc} (\mathbb{Z}) & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \prod_p \mathbb{Z}_p^{\wedge} & \longrightarrow & (\prod_p \mathbb{Z}_p^{\wedge}) \otimes \mathbb{Q} \end{array} \right)$$

Modules

- $$M \otimes \mathbb{Z}^{\square} = \begin{pmatrix} (M \otimes \mathbb{Z}) & \longrightarrow & M \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ M \otimes \prod_p \mathbb{Z}_p^{\wedge} & \longrightarrow & M \otimes (\prod_p \mathbb{Z}_p^{\wedge}) \otimes \mathbb{Q} \end{pmatrix}$$

- $$\otimes \mathbb{Z}^{\square} : \mathbb{Z}\text{-mod} \rightleftarrows \mathbb{Z}^{\square}\text{-mod} : Pull$$

Proposition

$$\mathbb{Z}\text{-}mod \simeq cell\text{-}\mathbb{Z}^{\square}\text{-}mod$$

$\mathbb{T}$-equivariant cohomology theories I

- $\mathbb{T}$-spectra, finite isotropy groups ($\mathbb{T}$=circle group)
- $R = \mathbb{S}$, $\mathcal{E}^{-1}\mathbb{S} = S^{\infty W(\mathbb{T})} = \bigcup_{W^{\mathbb{T}}=0} S^W = \tilde{E}\mathcal{F}$
- $\mathbb{S} \longrightarrow S^{\infty W(\mathbb{T})} \longrightarrow \Sigma E\mathcal{F}_+ \simeq \bigvee_n \Sigma E\langle n \rangle$
- $$\mathbb{S}^{\square} = \left(\begin{array}{ccc} (\mathbb{S}) & \longrightarrow & \tilde{E}\mathcal{F} \\ \downarrow & & \downarrow \\ F(E\mathcal{F}_+, \mathbb{S}) & \longrightarrow & \tilde{E}\mathcal{F} \wedge F(E\mathcal{F}_+, \mathbb{S}) \end{array} \right)$$

Proposition

$$\mathbb{S}\text{-}mod \simeq cell\text{-}\mathbb{S}^{\square}\text{-}mod$$

$\mathbb{T}$-equivariant cohomology theories II

- $$\mathbb{S}^{\Box} = \left(\begin{array}{ccc} (\mathbb{S}) & \longrightarrow & S^{\infty W(\mathbb{T})} \\ \downarrow & & \downarrow \\ \prod_n F(E\langle n\rangle, \mathbb{S}) & \longrightarrow & S^{\infty W(\mathbb{T})} \wedge \prod_n F(E\langle n\rangle, \mathbb{S}) \end{array} \right)$$
- Equivariant cohomology theories are built from cohomology of geometric $\mathbb{T}$-fixed points (i.e., $H^*(X^{\mathbb{T}})$ and Borel cohomology of C_n-fixed points (i.e., $H^*(E(\mathbb{T}/C_n)_+ \wedge_{\mathbb{T}/C_n} X^{C_n})$ for all n.
- $\pi_*^{\mathbb{T}}(S^{\infty W(\mathbb{T})}) = \mathbb{Q}$, $\pi_*^{\mathbb{T}}(F(E\langle n\rangle, \mathbb{S})) = H^*(B(\mathbb{T}/C_n)) = \mathbb{Q}[c]$

$\mathbb{T}$-equivariant cohomology theories III

- $\pi_*^{\mathbb{T}}(S^{\infty W(\mathbb{T})}) = \mathbb{Q}$, $\pi_*^{\mathbb{T}}(F(E\langle n\rangle, \mathbb{S})) = \mathbb{Q}[c]$
-
$$(\cdot)^{\square} = \left(\begin{array}{ccc} & & \mathbb{Q} \\ & & \downarrow \\ \prod_n \mathbb{Q}[c] & \longrightarrow & \mathcal{E}^{-1} \prod_n \mathbb{Q}[c] \end{array} \right)$$
- $\mathcal{A}(\mathbb{T})$ objects of $(\cdot)^{\square}$-mod so that horizontal and vertical are extensions of scalars

$$\begin{array}{ccc} & & V \\ & & \downarrow \\ N & \longrightarrow & P \end{array}$$

such that $P = \mathcal{E}^{-1}N$ and $P = \mathcal{E}^{-1}\prod_n \mathbb{Q}[c] \otimes V$.

The $\mathbb{T}^r$-sphere

- Repeat for the r-torus $G = \mathbb{T}^r$
- $$\mathbb{S}^{\square} = \left(\begin{array}{ccc} (\mathbb{S}) & \longrightarrow & \widetilde{E}\mathcal{F} \\ \downarrow & & \downarrow \\ F(E\mathcal{F}_+, \mathbb{S}) & \longrightarrow & \widetilde{E}\mathcal{F} \wedge F(E\mathcal{F}_+, \mathbb{S}) \end{array} \right)$$
- $F(E\mathcal{F}_+, \mathbb{S})$ is still algebraic (essentially $\prod_F H^*(BG/F)$)
- It remains to understand $\widetilde{E}\mathcal{F}$ ..... which can be done by induction on dimension and the inclusion-exclusion principle

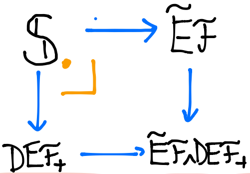
$\widetilde{E}\mathcal{F}$

- $\widetilde{E}\mathcal{F}$ has geometric isotropy $\widetilde{\mathcal{F}} = \{K \mid K \text{ not } \textit{finite}\}$ (a cofamily)
- We build it from simpler pieces, namely

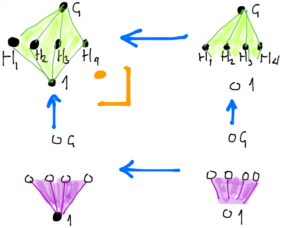
$$S^{\infty W(K)} = \bigcup_{W^K=0} S^W \text{ with geometric isotropy } V(K) = \{H \mid H \supseteq K\}$$

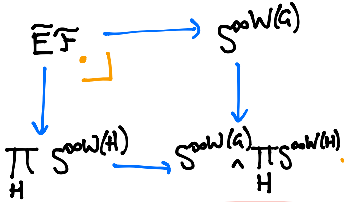
  a *principal* cofamily.
- In rank 1 $\widetilde{E}\mathcal{F} = S^{\infty W(G)}$ so this is easy
- In rank 2

$$\begin{array}{ccc} (\widetilde{E}\mathcal{F}) & \longrightarrow & S^{\infty W(G)} \\ \downarrow & & \downarrow \\ \prod_H S^{\infty W(H)} & \longrightarrow & S^{\infty W(G)} \wedge \prod_H S^{\infty W(H)} \end{array}$$

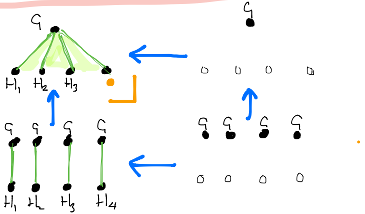


HASSE-TATE SQUARE





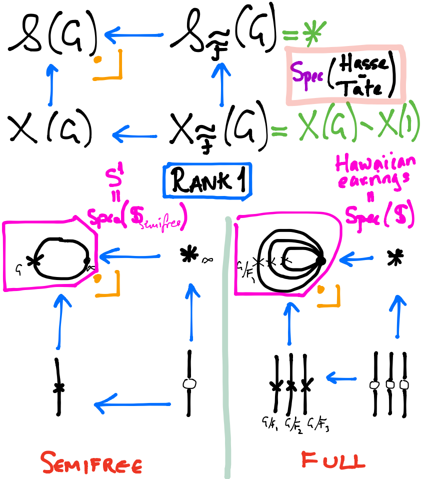
INCLUSION-EXCLUSION-ČECH

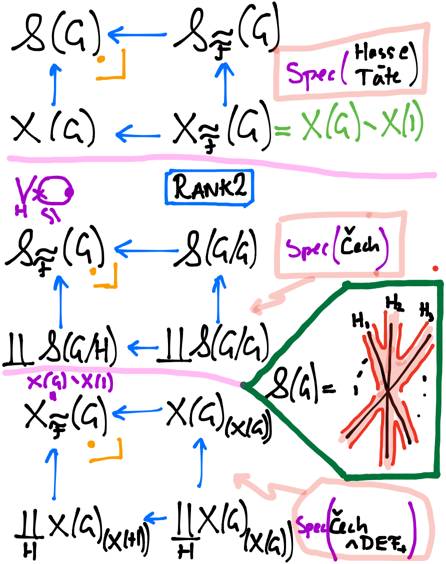


$$\mathcal{S}(G) = \text{"}\operatorname{Spec}(\mathbb{S})\text{"}$$

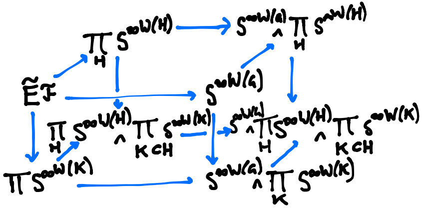
$$\begin{aligned} X(G) &= \operatorname{Spec}\Big(\prod_{\substack{F \\ \text{finite}}} H^{*}(BG/F)\Big) \\ &= \coprod_{F} \operatorname{Spec}\big(H^{\wedge}(BG/F)\big) \\ &= \coprod_{F} LG/F \end{aligned}$$

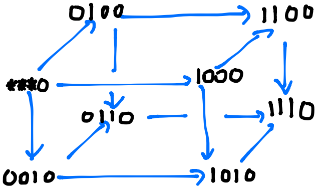
$$\operatorname{Spec}\left(\begin{array}{ccc} \mathbb{S} & \longrightarrow & \widetilde{E}\mathcal{F} \\ \downarrow & & \downarrow \\ D E\mathcal{F}_{+} & \longrightarrow & \widetilde{E}\mathcal{F} \wedge DE\mathcal{F}_{+} \end{array} \right)$$


S(G)
S_F̃(G) = *
Spec(Hasse-Tate)
X(G)
X_F̃(G) = X(G) \ X(1)
RANK 1
S¹ = Spec(S_semifree)
Hawaiian earrings = Spec(S)
G
∞
*∞
*
G/F₁
G/F₂
G/F₃
SEMIFREE
FULL


S(G) ← S_F̃(G)
Spec(Hasse–Tate)
X(G) ← X_F̃(G) = X(G) ∖ X(1)
RANK2
∀H
S_F̃(G) ← S(G/G)
Spec(Čech)
∐ S(G/H) ← ∐ S(G/G)
X(G) ∖ X(1)
X_F̃(G) ← X(G)_(X(G))
S(G) =
H₁ H₂ H₃
∐_H X(G)_(X(H)) ← ∐_H X(G)_(X(G))
Spec(Čech ∧DEF)

Rank 3





Calculating the homotopy of $\mathbb{S}$

- Organize the inductive construction to a single cube
- Find $\mathbb{S}$ as a pullback of *affine, formal* ring spectra
- Consider the poset of closed subgroups of G
- Consider the coefficient system $K \mapsto H^*(BG/K)$
- Add information on Euler classes
- Observe the homotopy of the ring spectra at the vertices comes from the coefficient system
- Define the associated Cech cohomology
- This calculates $\pi_*^G(\mathbb{S})$

Rank 1

In ring spectra we have

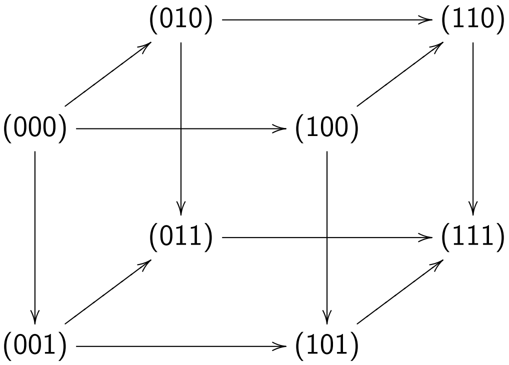
$$\begin{array}{ccc} & & S^{\infty W(G)} \\ & & \downarrow \\ DE\mathcal{F}_+ & \rightarrow & S^{\infty W(G)} \wedge DE\mathcal{F}_+ \end{array}$$

and in rings

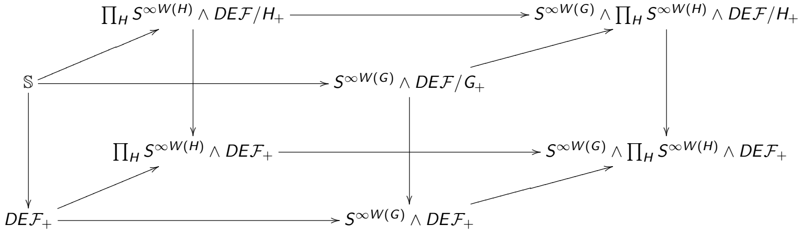
$$\begin{array}{ccc} & & \mathbb{Q} \\ & & \downarrow \\ \prod_F H^*(BG/F) & \rightarrow & \mathcal{E}_G^{-1} \prod_F H^*(BG/F) \end{array}$$

Rank 2

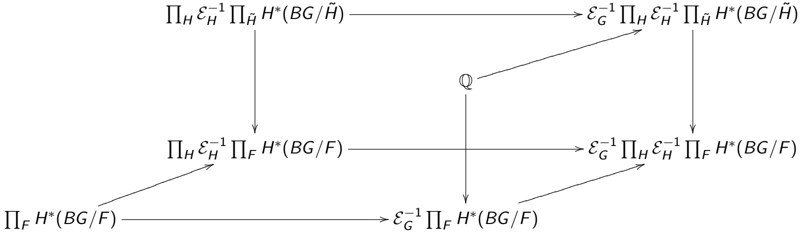
In rank 2, the layout of the vectors is



The diagram of ring spectra is as follows:



and the diagram of rings is as follows:



Cech cohomology

Associated to this covering we have

$$C^s_{cts}(\Sigma; R) = \prod_{(d_0 > \cdots > d_s)} C_{cts}^{(d_0 > \cdots > d_s)}(\Sigma; R)$$

$$C_{cts}^{(d_0 > \cdots > d_s)}(\Sigma; R) = \prod_{K_0} \mathcal{E}_{K_0}^{-1} \prod_{K_1 \subset K_0} \mathcal{E}_{K_1}^{-1} \cdots \mathcal{E}_{K_{s-1}}^{-1} \prod_{K_s \subset K_{s-1}} \prod_{\tilde{K}_s} H^*(BG/\tilde{K}_s)$$

$$= \text{continuous functions on } \coprod_{K_0 \supset K_1 \supset \cdots \supset K_s} \coprod_{\tilde{K}_s} X(G/\tilde{K}_s, \tilde{K}_0/\tilde{K}_s)$$

Corollary

There is a spectral sequence

$$H^*_{cts}(\Sigma, R) \Rightarrow \pi_*^G(\mathbb{S})$$

The end

Thank you for listening!