On the vanishing of negative equivariant $K$-theory

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Weibel’s conjecture on negative $K$-theory

**Conjecture (Weibel, 1980)**

Let $X$ be a Noetherian scheme of Krull dimension $d$. Then

$$K_i(X) = 0 \quad \text{for all} \quad i < -d.$$  

Equivalently, the spectrum $K^B(X)$ is $(-d)$-connective.

**Known cases:**

- $X$ is an excellent surface [Weibel, 2001]
- $X$ is essentially of finite type over a field of characteristic zero [Cortiñas–Haesemeyer–Schlichting–Weibel, 2008]
- $K_{<-d}(X)[1/p] = 0$ if $X$ is quasi-excellent and $p$ is nilpotent on $X$ [Kelly, 2014]
- $K_{<-d}(X)[1/n] = 0$ if $n$ is nilpotent on $X$,
  $K_{<-d}(X, \mathbb{Z}/n) = 0$ if $n$ is invertible on $X$ [Kerz–Strunk, 2016]
Equivariant $K$-theory

For $X$ an Artin stack, denote by:

- $D_{qcoh}(X) = \text{holim}_{\text{Spec}(A) \to X} D(A)$ the dg-category of quasi-coherent sheaves
- $D_{perf}(X) \subset D_{qcoh}(X)$ the subcategory of $\otimes$-dualizable objects

**Definition**

The $K$-theory spectrum $K^B(X)$ is the nonconnective $K$-theory of $D_{perf}(X)$, in the sense of Schlichting and Cisinski–Tabuada. For $i \in \mathbb{Z}$,

$$K_i(X) = \pi_i(K^B(X)).$$

**Example**

If $G$ is a linearly reductive group scheme over a field $F$, then $K_0(BG)$ is the representation ring of $G$ over $F$. 
The main theorem

**Theorem (H–Krishna)**

Let $S$ be a Noetherian scheme and $G$ an $S$-group scheme which is either locally diagonalizable or finite flat of degree invertible on $S$. Let $\mathcal{X} = [X/G]$ where $X$ is $G$-equivariantly quasi-projective over $S$.

- If $n$ is nilpotent on $X$, $K_i(\mathcal{X})[1/n] = 0$ for all $i < -\dim(X)$.
- If $n$ is invertible on $X$, $K_i(\mathcal{X}, \mathbb{Z}/n) = 0$ for all $i < -\dim(X)$.

**Remarks:**

- If $G$ is a finite discrete group, the quasi-projectivity assumption can be dropped.
- For a more canonical formulation, replace $\dim(X)$ by the cdh cohomological dimension of $\mathcal{X}$.
- Like Kerz–Strunk, we actually prove a stronger vanishing result about **homotopy K-theory**.
Let $X$ be a Noetherian scheme with $\text{dim}(X) = d$.

**Definition**

The **homotopy $K$-theory** spectrum of $X$ is

$$KH(X) = |K^B(\Delta^\bullet \times X)|.$$

**Theorem (Weibel)**

- $KH(X)[1/n] = K^B(X)[1/n]$ if $n$ is nilpotent on $X$.
- $KH(X, \mathbb{Z}/n) = K^B(X, \mathbb{Z}/n)$ if $n$ is invertible on $X$.

**Theorem (Kerz–Strunk)**

$KH(X)$ is $(-d)$-connective.
Let $\mathcal{X} = [X/G]$ be as in the main theorem, with $\dim(X) = d$.

**Definition**

The **homotopy $K$-theory** spectrum of $\mathcal{X}$ is

$$KH(\mathcal{X}) = |K^B(\Delta^\bullet \times \mathcal{X})|.$$ 

**Theorem (Krishna–Ravi)**

- $KH(\mathcal{X})[1/n] = K^B(\mathcal{X})[1/n]$ if $n$ is nilpotent on $\mathcal{X}$.
- $KH(\mathcal{X}, \mathbb{Z}/n) = K^B(\mathcal{X}, \mathbb{Z}/n)$ if $n$ is invertible on $\mathcal{X}$.

**Theorem (H–Krishna)**

$KH(\mathcal{X})$ is $(-d)$-connective.
The first main ingredient is **cdh descent** for $KH$:

**Theorem (Morel–Voevodsky, Ayoub, Cisinski)**

Let
- $X$ be a qcqs scheme,
- $Z \subset X$ a closed subscheme,
- $p : X' \to X$ an abstract blowup with center $Z$.

Then there is a long exact sequence

$$
\cdots \to KH_i(X) \to KH_i(Z) \oplus KH_i(X') \to KH_i(p^{-1}Z) \to KH_{i-1}(X) \to \cdots
$$

This is a formal consequence of:
- $KH(X)$ is the underlying spectrum of a **motivic spectrum** $KGL_X \in SH(X)$ such that, for every $f : Y \to X$, $f^*(KGL_X) \simeq KGL_Y$.
- the **gluing** and **proper base change** theorems in stable motivic homotopy theory.
The second main ingredient is **flatification by blowups**:

**Theorem (Gruson–Raynaud)**

Let $X$ be a Noetherian scheme, $U \subset X$ an open subset, $Y$ a finite type $X$-scheme, and $\mathcal{F}$ a finite type $\mathcal{O}_Y$-module flat over $U$. Then there exists a closed subscheme $Z \subset X \setminus U$ such that the strict transform of $\mathcal{F}$ is flat over $\text{Bl}_Z(X)$.

- By definition of negative $K$-theory, there is a surjective map

  $$\text{coker} \left( K_0(X \times \mathbb{A}^i) \to K_0(X \times \mathbb{G}_m^i) \right) \to K_{-i}(X), \quad i > 0. $$

- Using flatification, if $X$ is reduced with an ample family of line bundles, any negative $K$-theory class on $\Delta^k \times X$ can be killed by blowing up $X$.

- The vanishing of $KH_{<-d}(X)$ is then proved by induction on $d$ using cdh descent and the spectral sequence for the simplicial spectrum $K^B(\Delta^\bullet \times X)$. 

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Now for the equivariant story

- A suitable equivariant version of flatification by blowups was proved by Rydh.
- To complete the proof, following Kerz–Strunk, we need cdh descent for equivariant homotopy $K$-theory.

**Theorem (H)**

Let

- $\mathcal{X} = [X/G]$ be as in the main theorem,
- $\mathcal{Z} \subset \mathcal{X}$ a closed substack,
- $p: \mathcal{X}' \to \mathcal{X}$ an abstract blowup with center $\mathcal{Z}$.

Then there is a long exact sequence

$$
\cdots \to KH_i(\mathcal{X}) \to KH_i(\mathcal{Z}) \oplus KH_i(\mathcal{X}') \to KH_i(p^{-1}\mathcal{Z}) \to KH_{i-1}(\mathcal{X}) \to \cdots
$$

To prove this, we need stable equivariant motivic homotopy theory.
For a scheme $X$, $\text{SH}(X)$ is a $\otimes$-triangulated category built from smooth $X$-schemes. But smooth schemes can vary more generally in families parametrized by stacks, so it is reasonable to expect an extension

$$\{\text{schemes}\}^{\text{op}} \xrightarrow{\text{SH}} \{\otimes\text{-triangulated categories}\}.$$ 

This can be done for nice enough stacks, such as quotient stacks $[X/G]$ where $X$ is $G$-equivariantly quasi-projective over a base $S$ and $G$ is either

- finite locally free of degree invertible on $S$,
- of multiplicative type,
- arbitrary reductive if $S/\mathbb{Q}$. 

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The construction of $\text{SH}(\mathcal{X})$ is similar to the classical case, with a few tweaks:

- $\text{Sm}_{\mathcal{X}}$: smooth quasi-projective $\mathcal{X}$-stacks
- $\text{H}_{\cdot}(\mathcal{X})$: pointed presheaves of spaces $F$ on $\text{Sm}_{\mathcal{X}}$ satisfying:
  - **Nisnevich excision**: If $\mathcal{Y}' \to \mathcal{Y}$ is an étale neighborhood of a closed substack $\mathcal{Z} \subset \mathcal{Y}$, then $F(\mathcal{Y}) \simeq F(\mathcal{Y}') \times_{F(\mathcal{Y}' \setminus \mathcal{Z})} F(\mathcal{Y} \setminus \mathcal{Z})$.
  - **affine bundle invariance**: If $\mathcal{Y}' \to \mathcal{Y}$ is a torsor under a vector bundle, then $F(\mathcal{Y}) \simeq F(\mathcal{Y}')$.

- For every vector bundle $V$ over $\mathcal{X}$, let

  $$S^V = \mathbb{P}(V \oplus \mathbb{A}^1)/\mathbb{P}(V).$$

Then $\text{SH}(\mathcal{X})$ is the symmetric monoidal category obtained from $\text{H}_{\cdot}(\mathcal{X})$ by formally adjoining monoidal inverses $S^{-V}$. 
Theorem (H)

The extended $\text{SH}(\_)$ comes with **six operations**

$$f^* \dashv f_*, \quad f_! \dashv f^! \ (\text{for } f \text{ quasi-projective}), \quad \otimes \dashv \text{Hom},$$

with many expected properties.

In particular:

- **Gluing:** If $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed immersion with open complement $j : \mathcal{U} \hookrightarrow \mathcal{X}$, then $(i^*, j^*) : \text{SH}(\mathcal{X}) \to \text{SH}(\mathcal{Z}) \times \text{SH}(\mathcal{U})$ is comonadic.

- **Proper base change:** Given a cartesian square

$$\begin{array}{ccc}
\mathcal{Y}' & \xrightarrow{g} & \mathcal{X}' \\
q \downarrow & & \downarrow p \\
\mathcal{Y} & \xrightarrow{f} & \mathcal{X}
\end{array}$$

with $p$ projective, $f^* p_* \simeq q_* g^* : \text{SH}(\mathcal{X}') \to \text{SH}(\mathcal{Y})$. 
Theorem (H)

For $\mathcal{X} = [X/G]$ as in the main theorem, $KH(\mathcal{X})$ is the underlying spectrum of a motivic spectrum $KGL_{\mathcal{X}} \in SH(\mathcal{X})$. Moreover, for every $f : \mathcal{Y} \to \mathcal{X}$ quasi-projective, $f^*(KGL_{\mathcal{X}}) \simeq KGL_{\mathcal{Y}}$.

As before, this formally implies that $KH$ satisfies cdh descent, hence concludes the proof of the main theorem.

Sketch of proof:

- The existence of $KGL_{\mathcal{X}}$ follows from three basic properties of $KH$: homotopy invariance, Nisnevich descent, and Bott periodicity.

- The equivalence $f^*(KGL_{\mathcal{X}}) \simeq KGL_{\mathcal{Y}}$ is trivial if $f$ is smooth or if $\mathcal{X}$ is a scheme. To prove it in general, the idea is to find an explicit presentation of $KGL_{\mathcal{X}}$ in terms of infinite Grassmannians $\bigcup_{E/\mathcal{X}} \text{Gr}_n(E)$, which are stable under quasi-projective base change.
Thank you!