On the vanishing of negative equivariant K-theory

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Weibel's conjecture on negative K-theory

Conjecture (Weibel, 1980)

Let X be a Noetherian scheme of Krull dimension d. Then

$$K_i(X) = 0 \quad \text{for all} \quad i < -d.$$

Equivalently, the spectrum $K^B(X)$ is $(-d)$-connective.

Known cases:

- X is an excellent surface [Weibel, 2001]
- X is essentially of finite type over a field of characteristic zero [Cortiñas–Haesemeyer–Schlichting–Weibel, 2008]
- $K_{<-d}(X)[1/p] = 0$ if X is quasi-excellent and p is nilpotent on X [Kelly, 2014]
- $K_{<-d}(X)[1/n] = 0$ if n is nilpotent on X, $K_{<-d}(X, \mathbb{Z}/n) = 0$ if n is invertible on X [Kerz–Strunk, 2016]

Equivariant K-theory

For $\mathfrak{X}$ an Artin stack, denote by:

- $D_{\mathrm{qcoh}}(\mathfrak{X}) = \mathrm{holim}_{\mathrm{Spec}(A)\to\mathfrak{X}} D(A)$ the dg-category of quasi-coherent sheaves
- $D_{\mathrm{perf}}(\mathfrak{X}) \subset D_{\mathrm{qcoh}}(\mathfrak{X})$ the subcategory of $\otimes$-dualizable objects

Definition

The K-theory spectrum $K^B(\mathfrak{X})$ is the nonconnective K-theory of $D_{\mathrm{perf}}(\mathfrak{X})$, in the sense of Schlichting and Cisinski–Tabuada.
For $i \in \mathbb{Z}$,

$$K_i(\mathfrak{X}) = \pi_i(K^B(\mathfrak{X})).$$

Example

If G is a linearly reductive group scheme over a field F, then $K_0(BG)$ is the representation ring of G over F.

The main theorem

Theorem (H–Krishna)

Let S be a Noetherian scheme and G an S-group scheme which is either **locally diagonalizable** *or* **finite flat of degree invertible on** S.
Let $\mathfrak{X} = [X/G]$ where X is G-equivariantly quasi-projective over S.

- *If n is nilpotent on X, $K_i(\mathfrak{X})[1/n] = 0$ for all $i < -\dim(X)$.*
- *If n is invertible on X, $K_i(\mathfrak{X}, \mathbb{Z}/n) = 0$ for all $i < -\dim(X)$.*

Remarks:

- If G is a finite **discrete** group, the quasi-projectivity assumption can be dropped.
- For a more canonical formulation, replace $\dim(X)$ by the cdh cohomological dimension of $\mathfrak{X}$.
- Like Kerz–Strunk, we actually prove a stronger vanishing result about **homotopy K-theory**.

Homotopy K-theory

Let X be a Noetherian scheme with $\dim(X) = d$.

Definition

The **homotopy K-theory** spectrum of X is

$$KH(X) = |K^B(\Delta^\bullet \times X)|.$$

Theorem (Weibel)

$KH(X)[1/n] = K^B(X)[1/n]$ if n is nilpotent on X.
$KH(X, \mathbb{Z}/n) = K^B(X, \mathbb{Z}/n)$ if n is invertible on X.

Theorem (Kerz–Strunk)

$KH(X)$ is $(-d)$-connective.

Equivariant homotopy K-theory

Let $\mathfrak{X} = [X/G]$ be as in the main theorem, with $\dim(X) = d$.

Definition

The **homotopy K-theory** spectrum of $\mathfrak{X}$ is

$$KH(\mathfrak{X}) = |K^B(\Delta^\bullet \times \mathfrak{X})|.$$

Theorem (Krishna–Ravi)

$KH(\mathfrak{X})[1/n] = K^B(\mathfrak{X})[1/n]$ if n is nilpotent on $\mathfrak{X}$.
$KH(\mathfrak{X}, \mathbb{Z}/n) = K^B(\mathfrak{X}, \mathbb{Z}/n)$ if n is invertible on $\mathfrak{X}$.

Theorem (H–Krishna)

$KH(\mathfrak{X})$ is $(-d)$-connective.

Overview of the Kerz–Strunk proof, I

The first main ingredient is **cdh descent** for KH:

Theorem (Morel–Voevodsky, Ayoub, Cisinski)

Let

- X *be a qcqs scheme,*
- $Z \subset X$ *a closed subscheme,*
- $p\colon X' \to X$ *an abstract blowup with center* Z.

Then there is a long exact sequence

$$\cdots \to KH_i(X) \to KH_i(Z) \oplus KH_i(X') \to KH_i(p^{-1}Z) \to KH_{i-1}(X) \to \cdots$$

This is a formal consequence of:

- $KH(X)$ is the underlying spectrum of a **motivic spectrum** $\mathrm{KGL}_X \in \mathrm{SH}(X)$ such that, for every $f\colon Y \to X$, $f^*(\mathrm{KGL}_X) \simeq \mathrm{KGL}_Y$.
- the **gluing** and **proper base change** theorems in stable motivic homotopy theory.

Overview of the Kerz–Strunk proof, II

The second main ingredient is **flatification by blowups**:

Theorem (Gruson–Raynaud)

Let X be a Noetherian scheme, $U \subset X$ an open subset, Y a finite type X-scheme, and $\mathcal{F}$ a finite type $\mathcal{O}_Y$-module flat over U.
Then there exists a closed subscheme $Z \subset X \smallsetminus U$ such that the strict transform of $\mathcal{F}$ is flat over $\mathrm{Bl}_Z(X)$.

- By definition of negative K-theory, there is a surjective map

$$\mathrm{coker}\left(K_0(X \times \mathbb{A}^i) \to K_0(X \times \mathbb{G}_m^i)\right) \twoheadrightarrow K_{-i}(X), \quad i > 0.$$

- Using flatification, if X is reduced with an ample family of line bundles, any negative K-theory class on $\Delta^k \times X$ can be killed by blowing up X.
- The vanishing of $KH_{<-d}(X)$ is then proved by induction on d using cdh descent and the spectral sequence for the simplicial spectrum $K^B(\Delta^\bullet \times X)$.

Now for the equivariant story

- A suitable equivariant version of flatification by blowups was proved by Rydh.
- To complete the proof, following Kerz–Strunk, we need cdh descent for equivariant homotopy K-theory.

Theorem (H)

Let

- $\mathfrak{X} = [X/G]$ *be as in the main theorem,*
- $\mathfrak{Z} \subset \mathfrak{X}$ *a closed substack,*
- $p \colon \mathfrak{X}' \to \mathfrak{X}$ *an abstract blowup with center* $\mathfrak{Z}$*.*

Then there is a long exact sequence

$$\cdots \to KH_i(\mathfrak{X}) \to KH_i(\mathfrak{Z}) \oplus KH_i(\mathfrak{X}') \to KH_i(p^{-1}\mathfrak{Z}) \to KH_{i-1}(\mathfrak{X}) \to \cdots$$

To prove this, we need **stable equivariant motivic homotopy theory**.

Stable equivariant motivic homotopy theory, I

For a scheme X, $\mathrm{SH}(X)$ is a $\otimes$-triangulated category built from smooth X-schemes. But smooth schemes can vary more generally in families parametrized by **stacks**, so it is reasonable to expect an extension

$$\begin{tikzcd}
\{\text{schemes}\}^{\mathrm{op}} \arrow[r, "\mathrm{SH}"] \arrow[d, hook] & \{\otimes\text{-triangulated categories}\}. \\
\{\text{Artin stacks}\}^{\mathrm{op}} \arrow[ur, dashed] &
\end{tikzcd}$$

This can be done for nice enough stacks, such as quotient stacks $[X/G]$ where X is G-equivariantly quasi-projective over a base S and G is either

- finite locally free of degree invertible on S,
- of multiplicative type,
- arbitrary reductive if $S/\mathbb{Q}$.

Stable equivariant motivic homotopy theory, II

The construction of $\mathrm{SH}(\mathfrak{X})$ is similar to the classical case, with a few tweaks:

- $\mathrm{Sm}_{\mathfrak{X}}$: smooth quasi-projective $\mathfrak{X}$-stacks
- $\mathrm{H}_\bullet(\mathfrak{X})$: pointed presheaves of spaces F on $\mathrm{Sm}_{\mathfrak{X}}$ satisfying:
  – **Nisnevich excision**: If $\mathfrak{Y}' \to \mathfrak{Y}$ is an étale neighborhood of a closed substack $\mathfrak{Z} \subset \mathfrak{Y}$, then $F(\mathfrak{Y}) \simeq F(\mathfrak{Y}') \times_{F(\mathfrak{Y}' \smallsetminus \mathfrak{Z})} F(\mathfrak{Y} \smallsetminus \mathfrak{Z})$.
  – **affine bundle invariance**: If $\mathfrak{Y}' \to \mathfrak{Y}$ is a torsor under a vector bundle, then $F(\mathfrak{Y}) \simeq F(\mathfrak{Y}')$.
- For every vector bundle V over $\mathfrak{X}$, let

$$S^V = \mathbb{P}(V \oplus \mathbb{A}^1)/\mathbb{P}(V).$$

  Then $\mathrm{SH}(\mathfrak{X})$ is the symmetric monoidal category obtained from $\mathrm{H}_\bullet(\mathfrak{X})$ by formally adjoining monoidal inverses S^{-V}.

Theorem (H)

The extended SH$(-)$ *comes with* **six operations**

$$f^* \dashv f_*, \qquad f_! \dashv f^! \text{ (for } f \text{ quasi-projective)}, \qquad \otimes \dashv \mathrm{Hom},$$

with many expected properties.

In particular:

- **Gluing:** If $i\colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ is a closed immersion with open complement $j\colon \mathfrak{U} \hookrightarrow \mathfrak{X}$, then $(i^*, j^*)\colon \mathrm{SH}(\mathfrak{X}) \to \mathrm{SH}(\mathfrak{Z}) \times \mathrm{SH}(\mathfrak{U})$ is comonadic.
- **Proper base change:** Given a cartesian square

$$\begin{tikzcd} \mathfrak{Y}' \arrow[r, "g"] \arrow[d, "q"'] & \mathfrak{X}' \arrow[d, "p"] \\ \mathfrak{Y} \arrow[r, "f"'] & \mathfrak{X} \end{tikzcd}$$

with p projective, $f^*p_* \simeq q_*g^*\colon \mathrm{SH}(\mathfrak{X}') \to \mathrm{SH}(\mathfrak{Y})$.

Cdh descent for equivariant homotopy K-theory

Theorem (H)

For $\mathfrak{X} = [X/G]$ as in the main theorem, $KH(\mathfrak{X})$ is the underlying spectrum of a motivic spectrum $\mathrm{KGL}_{\mathfrak{X}} \in \mathrm{SH}(\mathfrak{X})$. Moreover, for every $f\colon \mathfrak{Y} \to \mathfrak{X}$ quasi-projective, $f^*(\mathrm{KGL}_{\mathfrak{X}}) \simeq \mathrm{KGL}_{\mathfrak{Y}}$.

As before, this formally implies that KH satisfies cdh descent, hence concludes the proof of the main theorem.

Sketch of proof:

- The existence of $\mathrm{KGL}_{\mathfrak{X}}$ follows from three basic properties of KH: **homotopy invariance**, **Nisnevich descent**, and **Bott periodicity**.
- The equivalence $f^*(\mathrm{KGL}_{\mathfrak{X}}) \simeq \mathrm{KGL}_{\mathfrak{Y}}$ is trivial if f is smooth or if $\mathfrak{X}$ is a scheme. To prove it in general, the idea is to find an explicit presentation of $\mathrm{KGL}_{\mathfrak{X}}$ in terms of infinite Grassmannians $\bigcup_{\mathcal{E}/\mathfrak{X}} \mathrm{Gr}_n(\mathcal{E})$, which are stable under quasi-projective base change.

Thank you!