$A^{1}$-homotopy invariance in spectral algebraic geometry

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Brave new algebra
Definition

An $\mathbf{E}_\infty$-ring spectrum is a commutative monoid in the symmetric monoidal $(\infty, 1)$-category of spectra.

Any $\mathbf{E}_\infty$-ring spectrum $R$ gives rise to a multiplicative generalized cohomology theory on spaces.
Examples of $E_\infty$-ring spectra

- The **sphere spectrum** $\mathbb{S}$ is the unit of the monoidal structure, and admits a canonical $E_\infty$-ring structure.
- For any ordinary commutative ring $R$, the **Eilenberg–MacLane spectrum** of $R$ is an $E_\infty$-ring spectrum (which we will denote again by $R$).
- The **complex K-theory spectrum** $KU$, and its connective version $ku$, are $E_\infty$-ring spectra.
Algebraic K-theory
Let $R$ be an $\mathbb{E}_\infty$-ring spectrum.

Write $K^{cn}(R)$ for the connective algebraic K-theory spectrum of $R$. Write $K(R)$ for the nonconnective K-theory spectrum of $R$, defined by the Bass construction.
Question

*How to compute the algebraic K-theory of the sphere spectrum?*

This is a very difficult problem with many potential applications. Ausoni–Rognes have some important conjectures about it.

We propose to study a variant of K-theory that we can say more about, and which is related to algebraic K-theory by a spectral sequence.
Let $R\{t\}$ denote the free $E_\infty$-$R$-algebra on one generator $t$. Explicitly:

$$R\{t\} = \bigoplus_{m \geq 0} R^\otimes m / \Sigma_m.$$ 

**Remark**

There is also a polynomial $E_\infty$-$R$-algebra $R[t]$, whose homotopy groups are as expected: $\pi_i(R[t]) = \pi_i(R)[t]$.

However, $R\{t\}$ and $R[t]$ look very different outside characteristic zero.
As $n$ varies, the $R$-algebras $R\{t\} \otimes^n$ can be given a canonical structure of simplicial object, which we denote $\Delta_{R, \bullet}$.

**Definition**

The **topological homotopy invariant K-theory spectrum of $R$** is defined as

$$TKH(R) := |K(\Delta_{R, \bullet})|,$$

i.e. the geometric realization of the simplicial spectrum $K(\Delta_{R, \bullet})$. 
TKH is the “brave new” analogue of Weibel’s homotopy invariant K-theory $KH$ of commutative rings.

The difference between $KH$ and $TKH$ is roughly analogous to the difference between $HH$ and $THH$. 
Nil-invariance of $TKH$

**Theorem (−)**

*Let $R$ be a connective $E_\infty$-ring spectrum. Then the canonical morphism of spectra

$$TKH(R) \xrightarrow{\sim} TKH(\pi_0(R))$$

is invertible.*

This is obtained as a consequence of a much stronger result about Morel–Voevodsky motivic homotopy theory in spectral algebraic geometry.

**Remark**

Nil-invariance also holds for another variant of $K$-theory, namely $G$-theory ($= K$-theory of coherent modules). This follows from C. Barwick’s Theorem of the Heart.
A spectral sequence

The skeletal filtration on the simplicial object $K(\Delta_R, \bullet)$ gives rise to:

**Proposition (-)**

*There is a right half-plane convergent spectral sequence*

$$E_1^{p,q} = N^{(p)}K_q(R) \Rightarrow TKH_{p+q}(R).$$

$N^{(p)}$ is defined iteratively with $N(K_q)$ denoting the functor $R \mapsto \text{Ker}(K_q(R[t]) \to K_q(R))$.

**Question**

*Does this degenerate with $\mathbb{Z}/\ell$ coefficients, when $\ell$ is invertible in $R$, so that*

$$K_q(R, \mathbb{Z}/\ell) = TKH_q(R, \mathbb{Z}/\ell)?$$
Spectral algebraic geometry (a.k.a. brave new algebraic geometry)
Spectral algebraic geometry is a version of algebraic geometry where the basic building blocks, commutative rings, are replaced by connective $E_\infty$-ring spectra.

Hence an *affine spectral scheme* is of the form $\text{Spec}(R)$ for some connective $E_\infty$-ring spectrum $R$.

Spectral schemes are then obtained by gluing affine spectral schemes along a natural notion of Zariski open immersion.
The underlying classical scheme

Every spectral scheme $S$ has an underlying classical scheme $S_{\text{cl}}$. For $S = \text{Spec}(R)$, $S_{\text{cl}} = \text{Spec}(\pi_0(R))$.

One thinks of $S$ as a nilpotent thickening of $S_{\text{cl}}$. 
Morel–Voevodsky homotopy theory
Let $S$ be a (classical) scheme. The $S^1$-stable motivic homotopy category $\text{SH}(S)$ is built out of smooth schemes over $S$ by imposing a Mayer–Vietoris condition and homotopy invariance with respect to the affine line $\mathbb{A}^1$. 
Étale and smooth morphisms in spectral AG

Given a morphism of connective $\mathbf{E}_\infty$-ring spectra $A \to B$, we have the **cotangent complex** $L_{B/A}$, a connective $B$-module, which controls the deformation theory of the $A$-algebra $B$.

**Definition**

*The morphism $A \to B$ is **étale** (TAQ-étale) if it is of finite presentation and $L_{B/A} = 0$.*

*The morphism $A \to B$ is **smooth** (TAQ-smooth) if it is of finite presentation and $L_{B/A}$ is a finitely generated projective module (i.e. a direct summand of $B^\oplus k$ for some $k$).*
Proposition (Toën–Vezzosi)
A morphism $A \to B$ is étale if and only if it is flat and induces an étale morphism $\pi_0(A) \to \pi_0(B)$ of commutative rings.

The analogue for smooth morphisms fails!

Example
The morphism of connective $E_\infty$-ring spectra $F_p \to F_p\{t\}$ is smooth, but $F_p \to F_p[t]$ is not.

Conceptually, the reason is the existence of nontrivial Steenrod operations in characteristic $p > 0$. 
Hence we have another version of smoothness for spectral schemes:

**Definition**

A morphism $A \to B$ is $♭$-smooth if it is flat, and induces a smooth morphism $\pi_0(A) \to \pi_0(B)$ of commutative rings.

We let $\text{Sm}_S$ denote the category of smooth spectral $S$-schemes.

We let $\text{Sm}^{♭}_S$ denote the category of $♭$-smooth spectral $S$-schemes.
Corresponding to our two notions of smoothness, we also have two affine lines.

**Definition**

*The affine line over $S = \text{Spec}(R)$ is the spectral scheme $\mathbb{A}^1_S = \text{Spec}(R\{t\})$.***

**Definition**

*The flat affine line over $S = \text{Spec}(R)$ is the spectral scheme $\mathbb{A}^1_{b,S} = \text{Spec}(R[t])$.***

$\mathbb{A}^1_S$ lives in $\text{Sm}_S$, while $\mathbb{A}^1_{b,S}$ lives in $\text{Sm}^b_S$. 
The two pairs $(\text{Sm}/_{S}, \mathbb{A}^{1}_{S})$ and $(\text{Sm}^{b}_{/S}, \mathbb{A}^{1}_{b,S})$ give rise to two different candidates for the motivic homotopy category over $S$:

$\text{SH}^{\text{brave}}(S) =$ brave new motivic homotopy category

$\text{SH}^{b}(S) =$ cowardly old motivic homotopy category

When $S$ is a classical scheme, $\text{SH}^{b}(S)$ recovers the usual $\text{SH}(S)$. 

We have the following analogue of an important theorem of Morel–Voevodsky in classical motivic homotopy theory:

**Theorem (−)**

Let $S$ be a spectral scheme, $i : Z \hookrightarrow S$ a closed immersion, and $j : U \hookrightarrow S$ the complementary open immersion. Then there is a short exact sequence of stable presentable $(\infty, 1)$-categories

$$\text{SH}^\text{brave}(Z) \to \text{SH}^\text{brave}(S) \to \text{SH}^\text{brave}(U).$$

I do not know whether this holds for the flat version $\text{SH}^\flat(S)$. 
Nil-invariance

The localization theorem has the following immediate consequence:

**Corollary**

*Let $S$ be a spectral scheme. Then there is a canonical equivalence of stable $(\infty, 1)$-categories*

\[ \mathcal{SH}^{\text{brave}}(S) \to \mathcal{SH}^{\text{brave}}(S_{\text{cl}}). \]
We return to our nil-invariance result for TKH:

**Theorem (-)**

*For each connective $E_\infty$-ring spectrum $R$, there is a canonical isomorphism of spectra*

$$TKH(R) \sim TKH(\pi_0(R)).$$

This theorem follows from the previous corollary, after proving that $TKH$ satisfies $A^1$-homotopy invariance and a Mayer–Vietoris condition, and that it is in fact representable by the group completion of the $A_\infty$-monoid $\biguplus_{n \geq 0} BGL_n$ (over quasi-compact quasi-separated spectral base schemes).
Generalized motivic cohomology theories

More generally, we obtain nil-invariance for any cohomology theory $E$ that is representable in $\mathbf{SH}^{\text{brave}}$ (as a cartesian section).
Questions

1. What is the analogue of the condition of regularity for $R$, that ensures $K(R) = TKH(R)$?

2. Is there a comparison isomorphism $TKH(R, \mathbb{Z}/\ell) = K(R, \mathbb{Z}/\ell)$ when $\ell$ is invertible in $R$?

3. Are there any $\mathbb{A}^1$-homotopy invariant cohomology theories of $E_{\infty}$-ring spectra that arise in nature?

4. Motivic cohomology of $E_{\infty}$-ring spectra?