# $\mathbf{A}^1$-homotopy invariance in spectral algebraic geometry

Adeel Khan
Universität Duisburg-Essen

Alpine Algebraic and Applied Topology Conference
Saas, 15–21 August 2016

# Brave new algebra

# $\mathbf{E}_\infty$-ring spectra

**Definition**

*An* $\mathbf{E}_\infty$**-ring spectrum** *is a commutative monoid in the symmetric monoidal* $(\infty, 1)$*-category of spectra.*

Any $\mathbf{E}_\infty$-ring spectrum $R$ gives rise to a *multiplicative* generalized cohomology theory on spaces.

# Examples of $\mathbf{E}_\infty$-ring spectra

- The sphere spectrum $\mathbb{S}$ is the unit of the monoidal structure, and admits a canonical $\mathbf{E}_\infty$-ring structure.
- For any ordinary commutative ring $R$, the Eilenberg–MacLane spectrum of $R$ is an $\mathbf{E}_\infty$-ring spectrum (which we will denote again by $R$).
- The complex K-theory spectrum $KU$, and its connective version $ku$, are $\mathbf{E}_\infty$-ring spectra.

# Algebraic K-theory

# Algebraic K-theory

Let $R$ be an $\mathbf{E}_\infty$-ring spectrum.

Write $K^{\mathrm{cn}}(R)$ for the connective algebraic K-theory spectrum of $R$.

Write $K(R)$ for the nonconnective K-theory spectrum of $R$, defined by the Bass construction.

# $K(\mathbb{S})$

**Question**

*How to compute the algebraic K-theory of the sphere spectrum?*

This is a very difficult problem with many potential applications. Ausoni–Rognes have some important conjectures about it.

We propose to study a variant of K-theory that we can say more about, and which is related to algebraic K-theory by a spectral sequence.

# Topological homotopy invariant K-theory

Let $R\{t\}$ denote the free $\mathbf{E}_\infty$-$R$-algebra on one generator $t$. Explicitly:

$$R\{t\} = \bigoplus_{m \geqslant 0} R^{\otimes m}/\Sigma_m.$$

**Remark**

There is also a polynomial $\mathbf{E}_\infty$-$R$-algebra $R[t]$, whose homotopy groups are as expected: $\pi_i(R[t]) = \pi_i(R)[t]$.

However, $R\{t\}$ and $R[t]$ look very different outside characteristic zero.

# Topological homotopy invariant K-theory

As $n$ varies, the $R$-algebras $R\{t\}^{\otimes n}$ can be given a canonical structure of simplicial object, which we denote $\Delta_{R,\bullet}$.

**Definition**

*The* **topological homotopy invariant K-theory** *spectrum of $R$ is defined as*

$$TKH(R) := |K(\Delta_{R,\bullet})|,$$

*i.e. the geometric realization of the simplicial spectrum $K(\Delta_{R,\bullet})$.*

# Topological homotopy invariant K-theory

$TKH$ is the "brave new" analogue of Weibel's homotopy invariant K-theory $KH$ of commutative rings.

The difference between $KH$ and $TKH$ is roughly analogous to the difference between $HH$ and $THH$.

# Nil-invariance of *TKH*

**Theorem (-)**

*Let $R$ be a connective $\mathbf{E}_\infty$-ring spectrum. Then the canonical morphism of spectra*

$$TKH(R) \xrightarrow{\sim} TKH(\pi_0(R))$$

*is invertible.*

This is obtained as a consequence of a much stronger result about Morel–Voevodsky motivic homotopy theory in spectral algebraic geometry.

**Remark**

Nil-invariance also holds for another variant of K-theory, namely $G$-theory ($=$ K-theory of coherent modules). This follows from C. Barwick's Theorem of the Heart.

## A spectral sequence

The skeletal filtration on the simplicial object $K(\Delta_{R,\bullet})$ gives rise to:

**Proposition (-)**

*There is a right half-plane convergent spectral sequence*

$$E_1^{p,q} = N^{(p)}K_q(R) \Rightarrow TKH_{p+q}(R).$$

$N^{(p)}$ is defined iteratively with $N(K_q)$ denoting the functor $R \mapsto \mathrm{Ker}(K_q(R\{t\}) \to K_q(R))$.

**Question**

*Does this degenerate with $\mathbf{Z}/\ell$ coefficients, when $\ell$ is invertible in $R$, so that*

$$K_q(R, \mathbf{Z}/\ell) = TKH_q(R, \mathbf{Z}/\ell)?$$

# Spectral algebraic geometry (a.k.a. brave new algebraic geometry)

# Spectral schemes

Spectral algebraic geometry is a version of algebraic geometry where the basic building blocks, commutative rings, are replaced by connective $\mathbf{E}_\infty$-ring spectra.

Hence an *affine spectral scheme* is of the form $\mathrm{Spec}(R)$ for some connective $\mathbf{E}_\infty$-ring spectrum $R$.

Spectral schemes are then obtained by gluing affine spectral schemes along a natural notion of Zariski open immersion.

# The underlying classical scheme

Every spectral scheme $S$ has an underlying classical scheme $S_{\mathrm{cl}}$.

For $S = \mathrm{Spec}(R)$, $S_{\mathrm{cl}} = \mathrm{Spec}(\pi_0(R))$.

One thinks of $S$ as a *nilpotent thickening* of $S_{\mathrm{cl}}$.

# Morel–Voevodsky homotopy theory

# The classical motivic homotopy category

Let $S$ be a (classical) scheme. The $S^1$-stable motivic homotopy category $\mathbf{SH}(S)$ is built out of smooth schemes over $S$ by imposing a Mayer–Vietoris condition and homotopy invariance with respect to the affine line $\mathbf{A}^1$.

# Étale and smooth morphisms in spectral AG

Given a morphism of connective $\mathbf{E}_\infty$-ring spectra $A \to B$, we have the **cotangent complex** $\mathbf{L}_{B/A}$, a connective $B$-module, which controls the deformation theory of the $A$-algebra $B$.

**Definition**

*The morphism $A \to B$ is* **étale** *(TAQ-étale) if it is of finite presentation and $\mathbf{L}_{B/A} = 0$.*

*The morphism $A \to B$ is* **smooth** *(TAQ-smooth) if it is of finite presentation and $\mathbf{L}_{B/A}$ is a finitely generated projective module (i.e. a direct summand of $B^{\oplus k}$ for some $k$).*

# Étale and smooth morphisms in spectral AG

**Proposition (Toën–Vezzosi)**

*A morphism $A \to B$ is étale if and only if it is flat and induces an étale morphism $\pi_0(A) \to \pi_0(B)$ of commutative rings.*

The analogue for smooth morphisms fails!

**Example**

*The morphism of connective $\mathbf{E}_\infty$-ring spectra $\mathbf{F}_p \to \mathbf{F}_p\{t\}$ is smooth, but $\mathbf{F}_p \to \mathbf{F}_p[t]$ is not.*

Conceptually, the reason is the existence of nontrivial Steenrod operations in characteristic $p > 0$.

# Étale and smooth morphisms in spectral AG

Hence we have another version of smoothness for spectral schemes:

**Definition**

*A morphism $A \to B$ is $\flat$-**smooth** if it is flat, and induces a smooth morphism $\pi_0(A) \to \pi_0(B)$ of commutative rings.*

We let $\mathrm{Sm}_{/S}$ denote the category of smooth spectral $S$-schemes.

We let $\mathrm{Sm}^{\flat}_{/S}$ denote the category of $\flat$-smooth spectral $S$-schemes.

# Homotopy invariance in spectral AG

Corresponding to our two notions of smoothness, we also have two affine lines.

**Definition**

*The* **affine line** *over* $S = \mathrm{Spec}(R)$ *is the spectral scheme* $\mathbf{A}^1_S = \mathrm{Spec}(R\{t\})$*.*

**Definition**

*The* **flat affine line** *over* $S = \mathrm{Spec}(R)$ *is the spectral scheme* $\mathbf{A}^1_{\flat,S} = \mathrm{Spec}(R[t])$*.*

$\mathbf{A}^1_S$ lives in $\mathrm{Sm}_{/S}$, while $\mathbf{A}^1_{\flat,S}$ lives in $\mathrm{Sm}^{\flat}_{/S}$.

# Motivic homotopy theory in spectral AG

The two pairs $(\mathrm{Sm}_{/S}, \mathbf{A}^1_S)$ and $(\mathrm{Sm}^\flat_{/S}, \mathbf{A}^1_{\flat,S})$ give rise to two different candidates for the motivic homotopy category over $S$:

$\mathbf{SH}^{\mathrm{brave}}(S) =$ brave new motivic homotopy category

$\mathbf{SH}^\flat(S) =$ cowardly old motivic homotopy category

When $S$ is a classical scheme, $\mathbf{SH}^\flat(S)$ recovers the usual $\mathbf{SH}(S)$.

# The localization theorem

We have the following analogue of an important theorem of Morel–Voevodsky in classical motivic homotopy theory:

**Theorem (-)**
*Let $S$ be a spectral scheme, $i : Z \hookrightarrow S$ a closed immersion, and $j : U \hookrightarrow S$ the complementary open immersion. Then there is a short exact sequence of stable presentable $(\infty, 1)$-categories*

$$\mathbf{SH}^{\mathrm{brave}}(Z) \to \mathbf{SH}^{\mathrm{brave}}(S) \to \mathbf{SH}^{\mathrm{brave}}(U).$$

I do not know whether this holds for the flat version $\mathbf{SH}^{\flat}(S)$.

# Nil-invariance

The localization theorem has the following immediate consequence:

**Corollary**

*Let $S$ be a spectral scheme. Then there is a canonical equivalence of stable $(\infty, 1)$-categories*

$$\mathbf{SH}^{\mathrm{brave}}(S) \to \mathbf{SH}^{\mathrm{brave}}(S_{\mathrm{cl}}).$$

We return to our nil-invariance result for $TKH$:

**Theorem (-)**
*For each connective $\mathbf{E}_\infty$-ring spectrum $R$, there is a canonical isomorphism of spectra*

$$TKH(R) \xrightarrow{\sim} TKH(\pi_0(R)).$$

This theorem follows from the previous corollary, after proving that $TKH$ satisfies $\mathbf{A}^1$-homotopy invariance and a Mayer–Vietoris condition, and that it is in fact representable by the group completion of the $\mathbf{A}_\infty$-monoid $\sqcup_{n\geqslant 0} BGL_n$ (over quasi-compact quasi-separated spectral base schemes).

# Generalized motivic cohomology theories

More generally, we obtain nil-invariance for any cohomology theory $E$ that is representable in $\mathbf{SH}^{\mathrm{brave}}$ (as a cartesian section).

## Questions

1. What is the analogue of the condition of regularity for $R$, that ensures $K(R) = TKH(R)$?
2. Is there is a comparison isomorphism $TKH(R, \mathbf{Z}/\ell) = K(R, \mathbf{Z}/\ell)$ when $\ell$ is invertible in $R$?
3. Are there any $\mathbf{A}^1$-homotopy invariant cohomology theories of $\mathbf{E}_\infty$-ring spectra that arise in nature?
4. Motivic cohomology of $\mathbf{E}_\infty$-ring spectra?