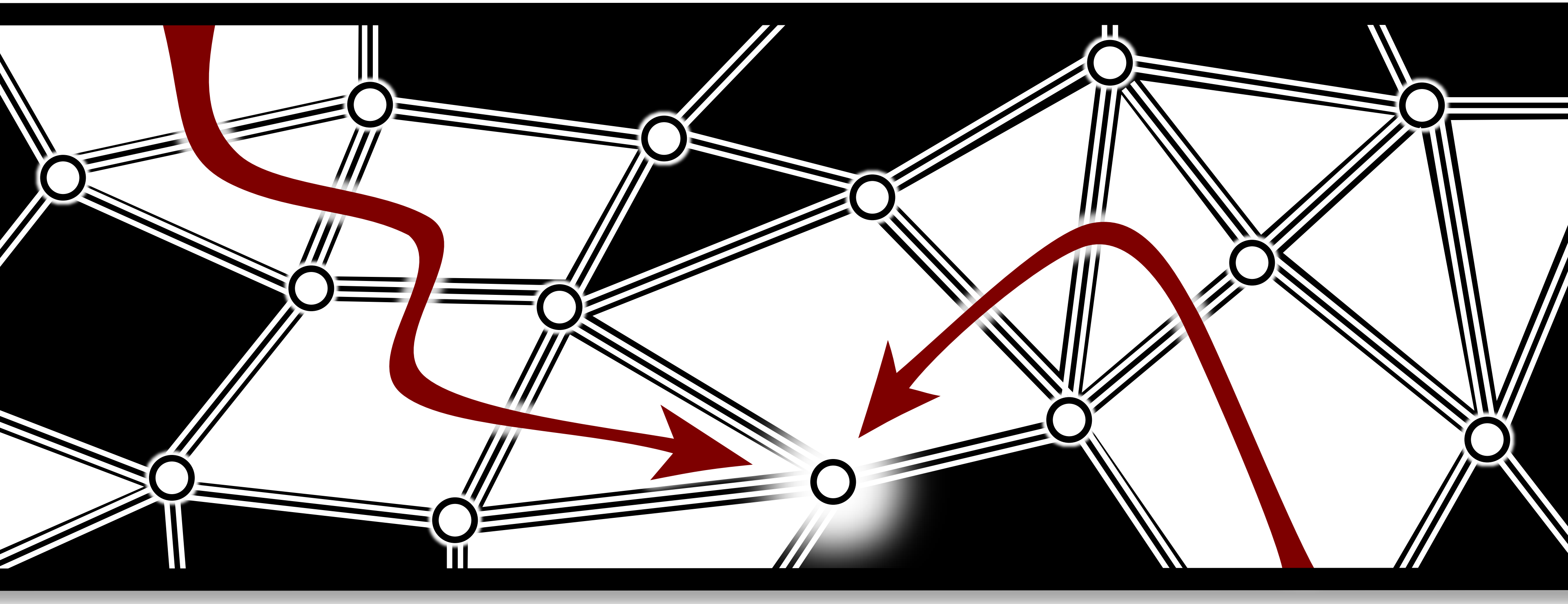
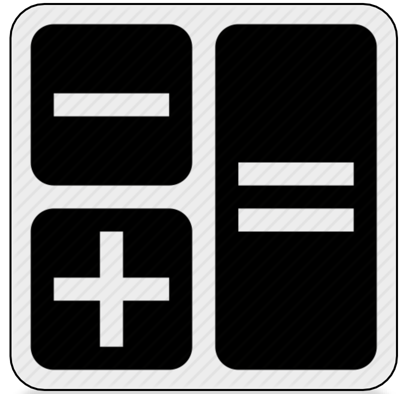
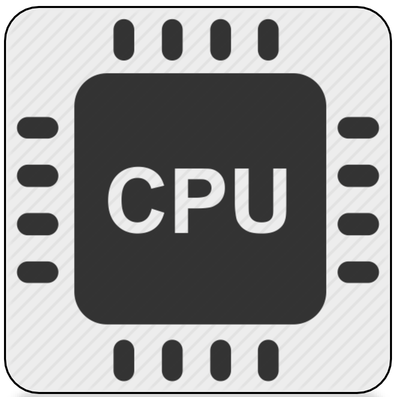
The Discrete Flow Category

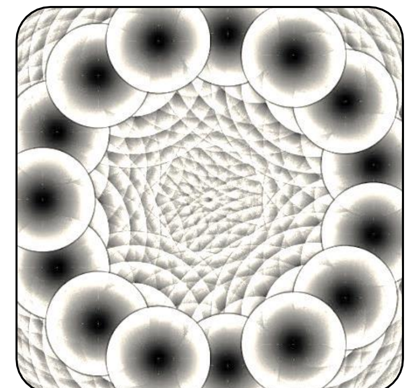


Vidit Nanda
The University of Pennsylvania
19/8/2016 @ Saas

Applied algebraic topology



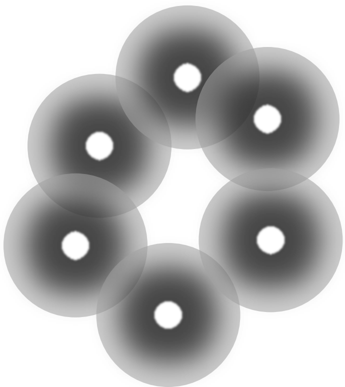




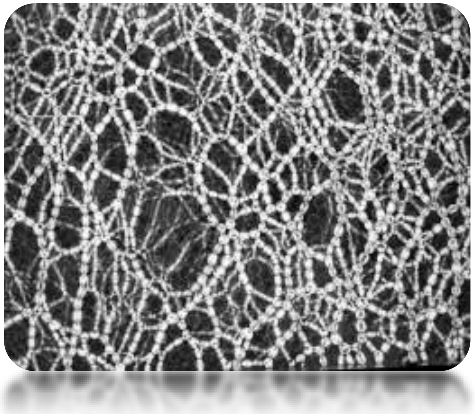
Outline

Topology for data

Algebraic-topological invariants (mainly homology) have found recent use in various scientific and engineering applications:



Sensor network coverage
(de Silva, Ghrist, Carlsson…)



Granular force chains
(Mischaikow, Kramar, …)

Compressibility prediction
(Hiraoka, Mischaikow, N…)

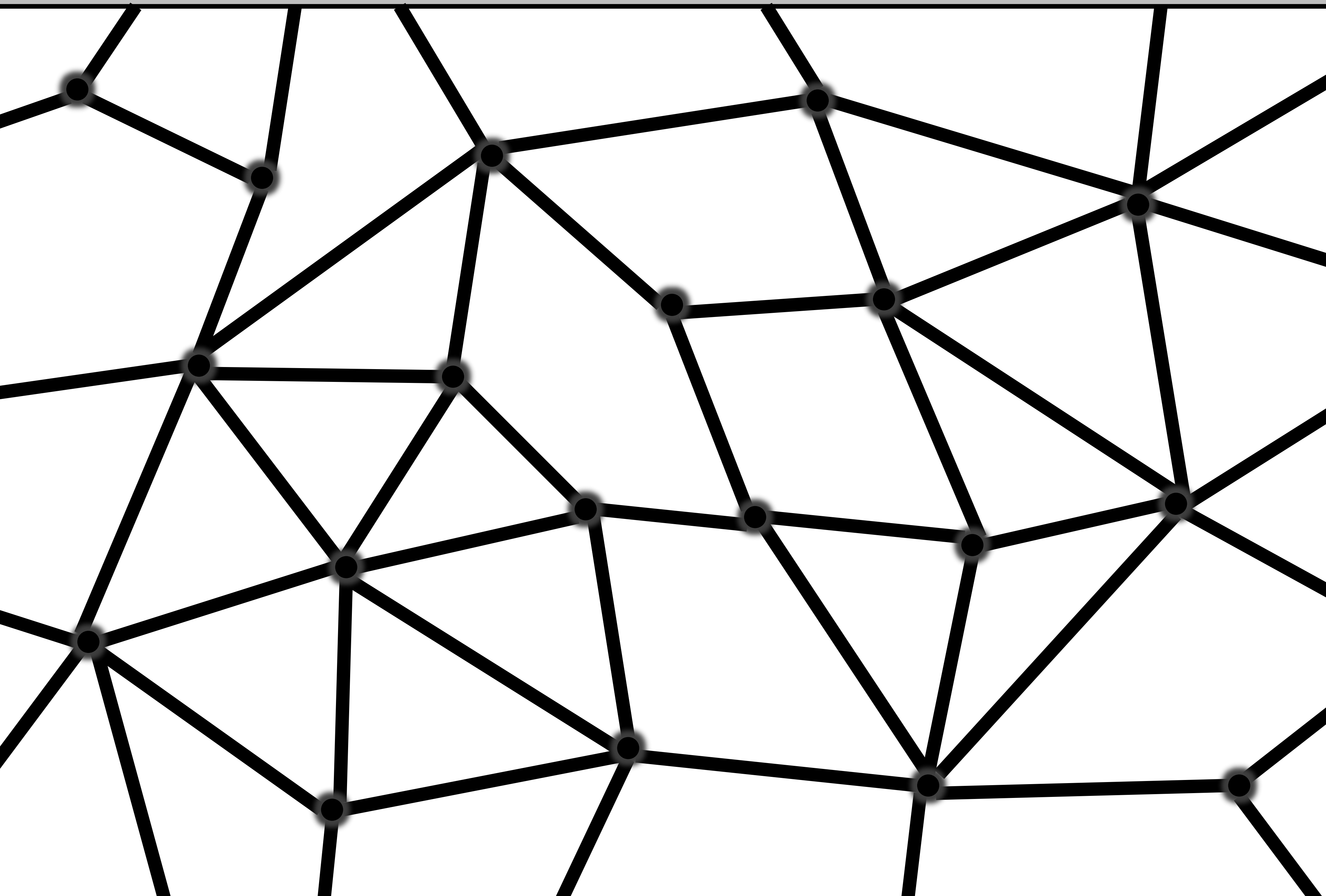
In each case, one builds a (filtered) cell complex around a point cloud, where d-cells correspond to (d+1)-fold intersections

Computing homology of such a cell complex reduces to linear algebraic row and column operations, and has cubic complexity in the total number of cells

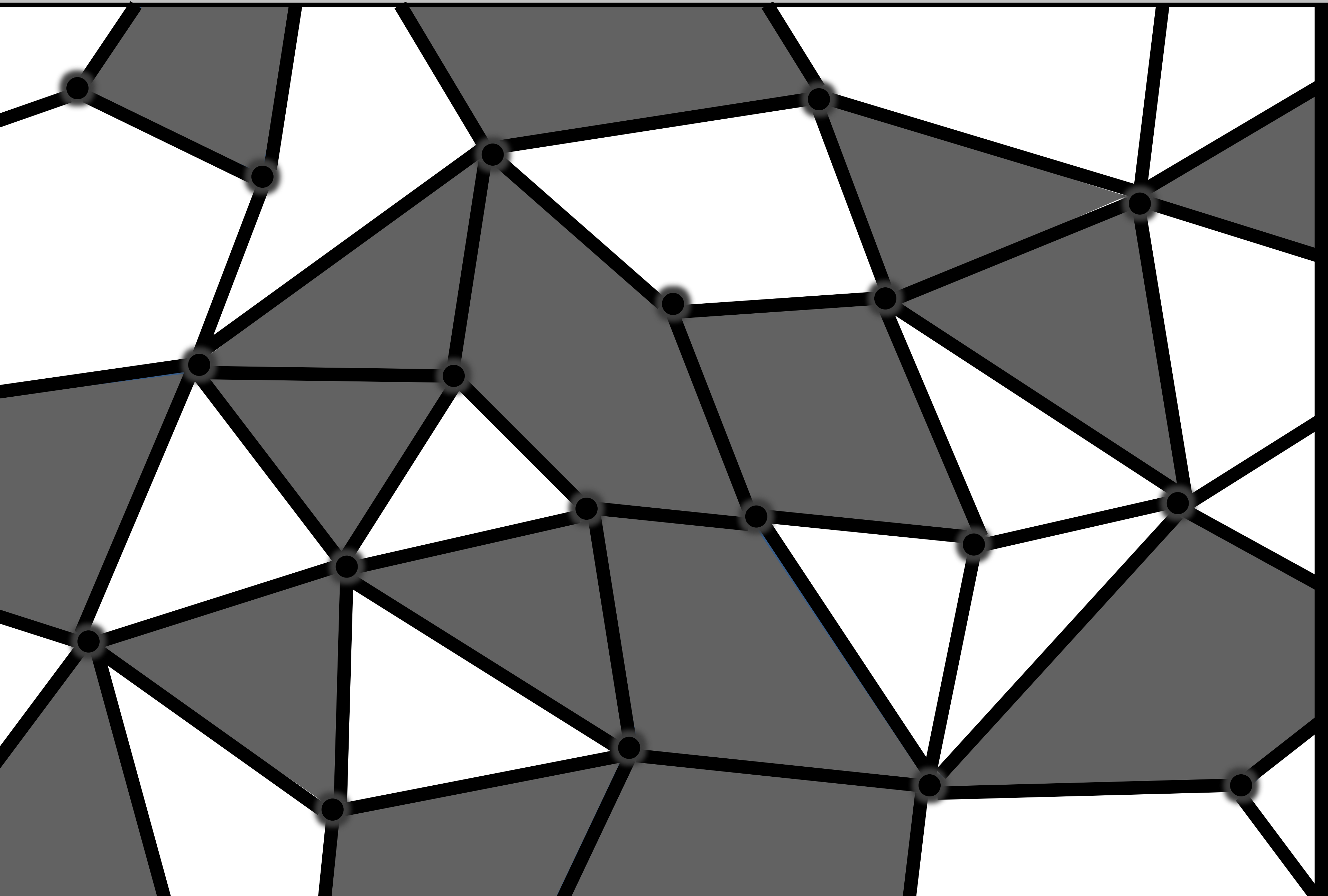
This is a **serious computational obstacle**

Why so serious?

Why so serious?



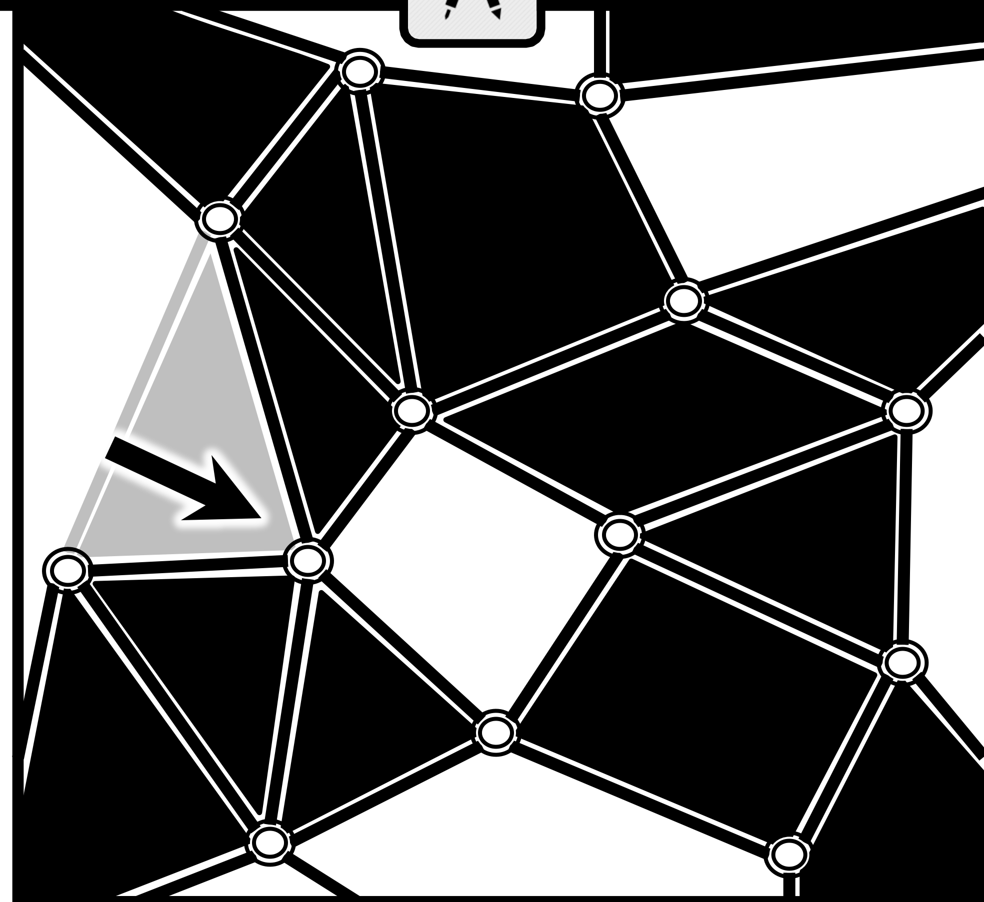
Why so serious?



"A pair of star-cross'd lovers..."

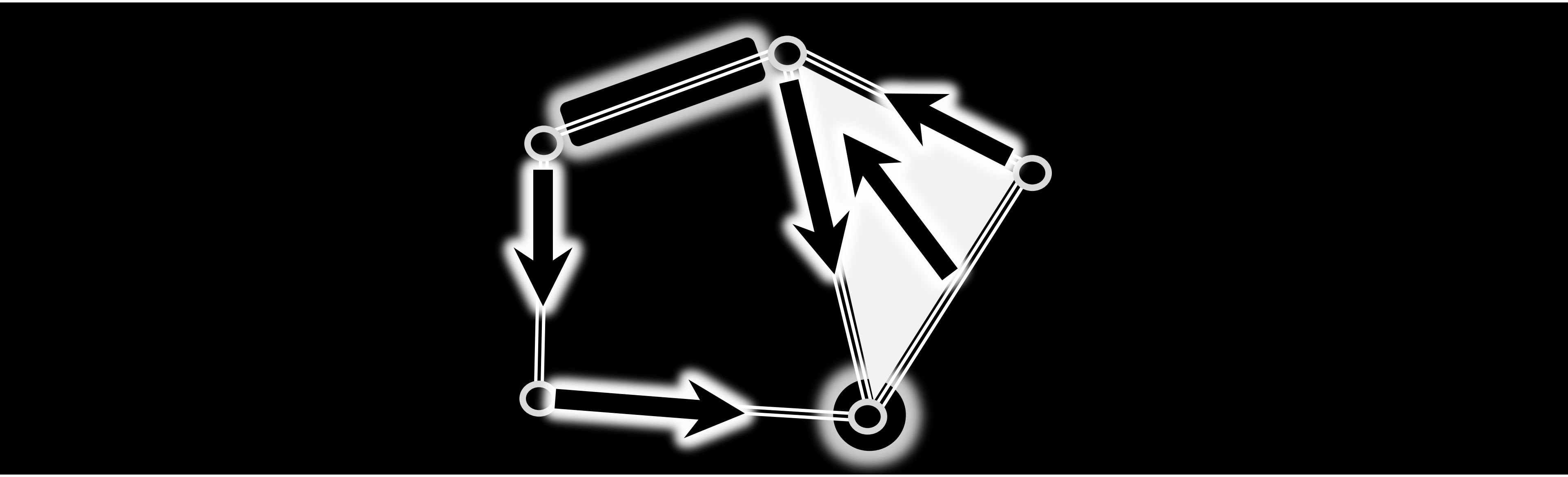
Removing a cell along with a **free face** preserves simple homotopy type (and hence, cellular homology with R-coefficients) provided that the degree of the attaching map is a unit in R (JHC Whitehead, 49)

$$\begin{bmatrix} \bullet & \bullet & \cdots & \bullet & \cdots & \bullet & \bullet \\ \vdots & \vdots & \cdots & \bullet & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \bullet & \cdots & \vdots & \vdots \\ \bullet & \bullet & \cdots & \bullet & \cdots & \bullet & \bullet \end{bmatrix}$$



Re-Morse theory

Discrete Morse theory (R Forman, 98) operates on CW complexes by partitioning the cells as *critical* or as *pairs* $\{(x_\bullet > y_\bullet)\}$



The *gradient vector field* flows from a cell to the cells in its boundary, *except* it flows against dimension along cell pairs

Thus, the paired cells must satisfy an acyclicity condition: no loops of the form

$$y_0 < x_0 > y_1 < x_1 > \cdots > y_0 < x_0$$

Alternately, the relation $(x > y) \blacktriangleright (x' > y')$ whenever $x > y'$ generates a partial order on cell pairs

Homology vs homotopy

Homology:

Let $\mathcal{F}$ be a constructible cosheaf (locally constant coefficient system taking values in an Abelian category) over a finite cell complex X

Let $\{(x_\bullet > y_\bullet)\}$ be a Morse pairing on cells so that every co-restriction map $\mathcal{F}(x_\bullet > y_\bullet)$ is an isomorphism in the target category

There is a **Morse chain complex** whose chain groups are direct sums of $\mathcal{F}(c)$ over critical cells, while the boundary operator comes from zigzags paths:

$$c > y_0 < x_0 > y_1 < x_1 > \cdots > y_n < x_n > c'$$

Namely, assign to each such path the weight

$$\mathcal{F}(x_n > c') \circ \mathcal{F}(x_n > y_n)^{-1} \circ \cdots \circ \mathcal{F}(x_0 > y_0)^{-1} \circ \mathcal{F}(c > y_0)$$

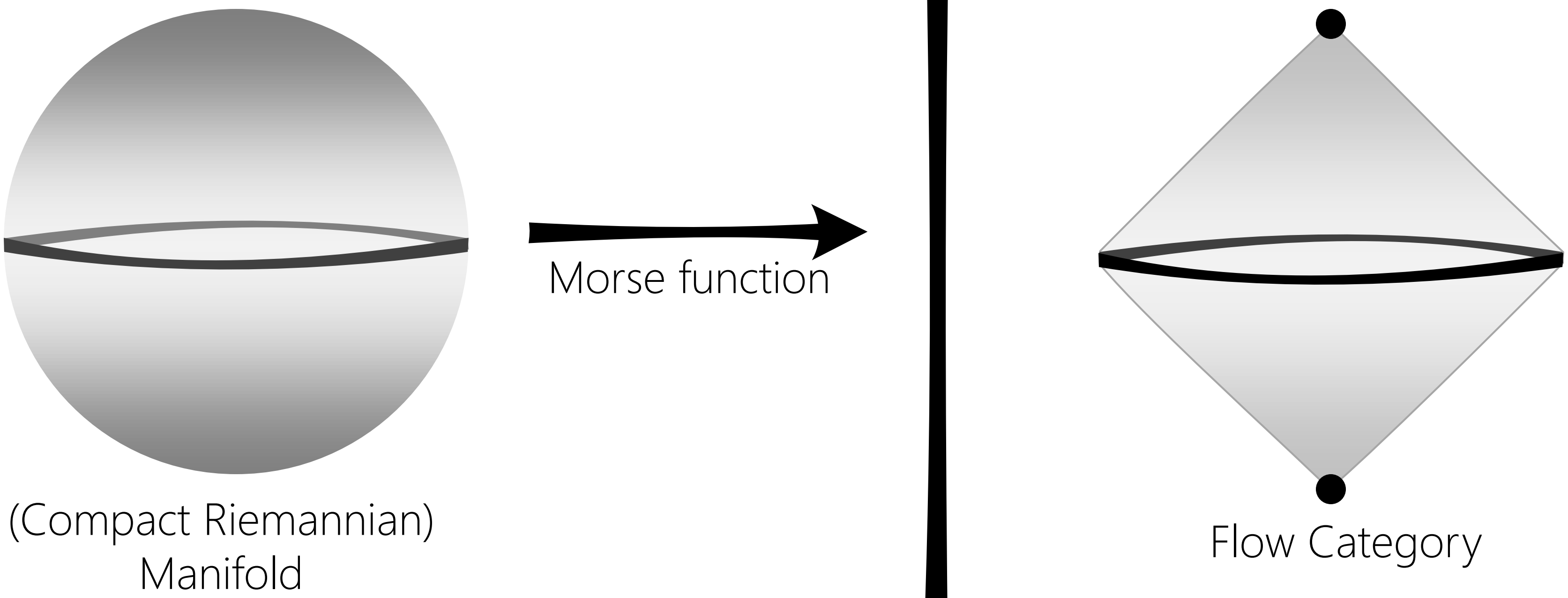
and use the sum-of-weights-over-paths

The Morse chain complex is explicitly quasi-isomorphic to the original one, and hence recovers the homology $H_\bullet(X; \mathcal{F})$

Homotopy:

We know that the original complex is homotopy-equivalent to a new one built with only the critical cells, but *we don't know the actual attaching maps*

The smooth flow category



[R Cohen, J Jones, G Segal, 95]

There is a (topologically enriched) **flow category** whose:

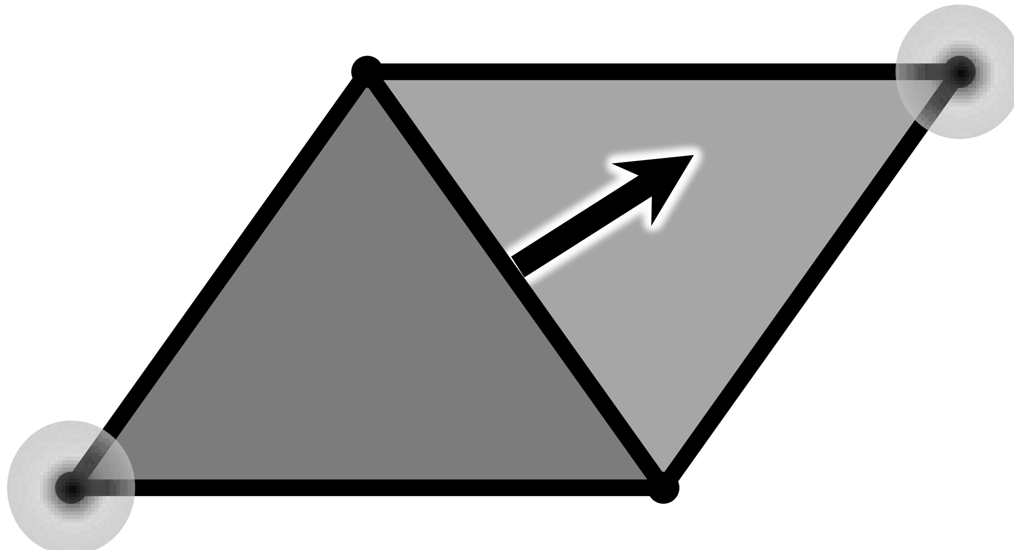
objects are critical points of the Morse function, and

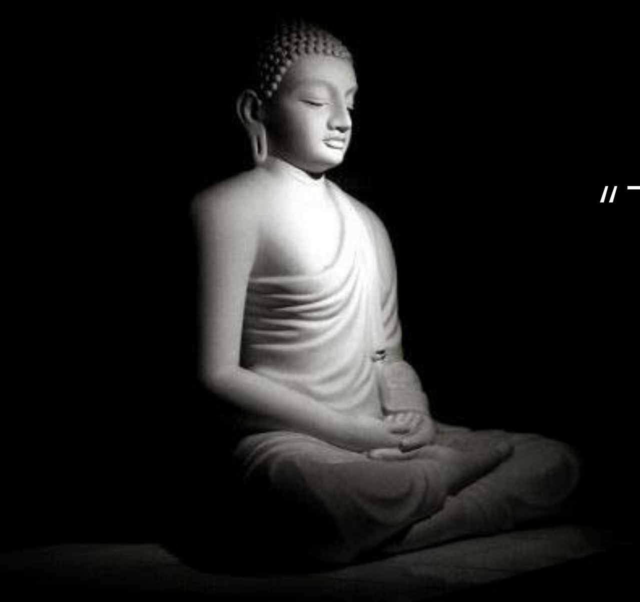
morphisms are moduli spaces of gradient trajectories,

and whose classifying space is homotopy-equivalent to the manifold

Desideratum: a discrete, computable analogue *for cell complexes*

Over-attachment

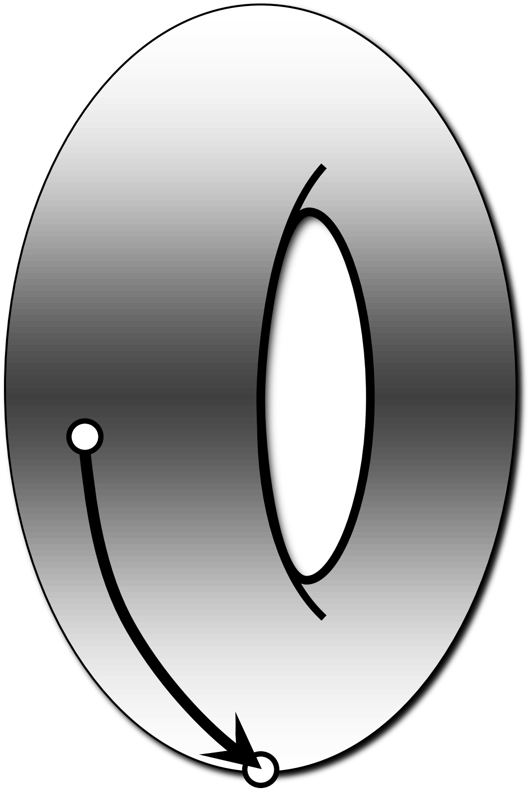




"The root of all suffering is attachment."

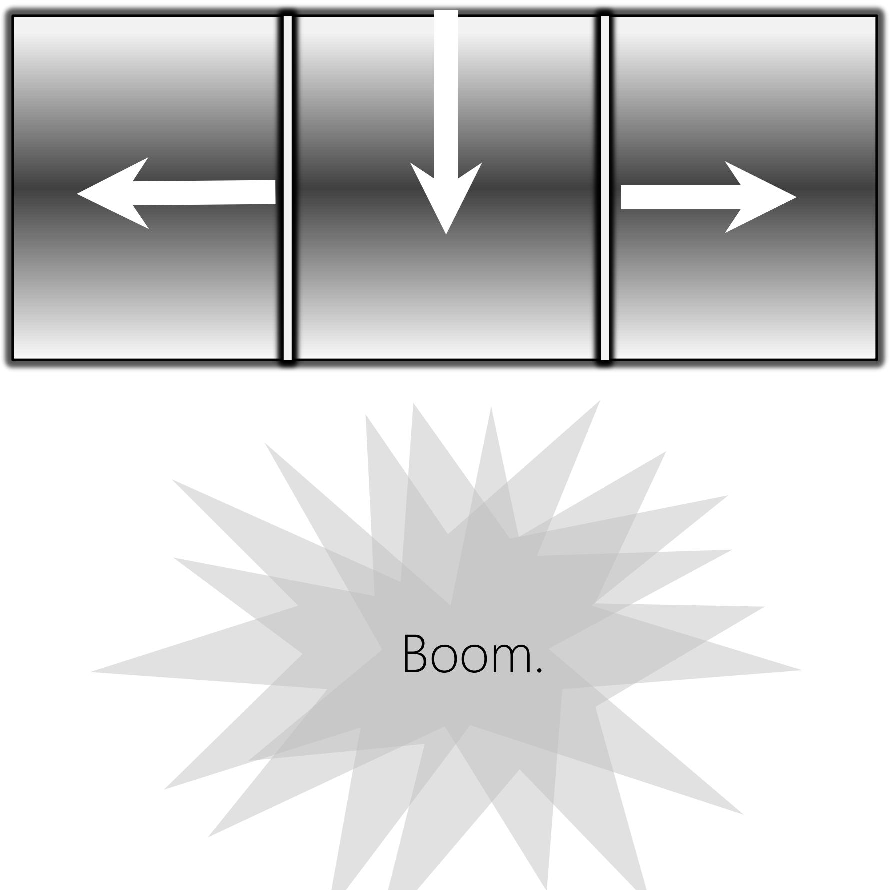
Lost in translation

Smooth Flow



Every non-critical point flows via some *unique* gradient path to a *single* critical cell of the smooth Morse function

Discrete Flow

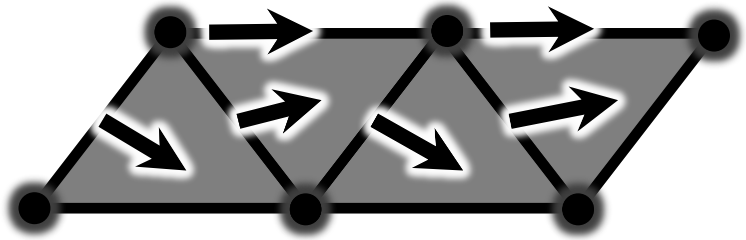


Need something new…

Smooth	Discrete
Compact Riemannian Manifold	Regular Cell Complex
Morse function	Cell pairing
Index k critical points	k-dimensional unpaired cells
Gradient flow lines	Zigzags paths of cells
Moduli spaces of flow lines	?????????????????

We must impose a topology on the set of all zigzag paths between a fixed pair of critical cells

The simplest thing to try is a partial order: when is a zigzag path “less than” another zigzag path?

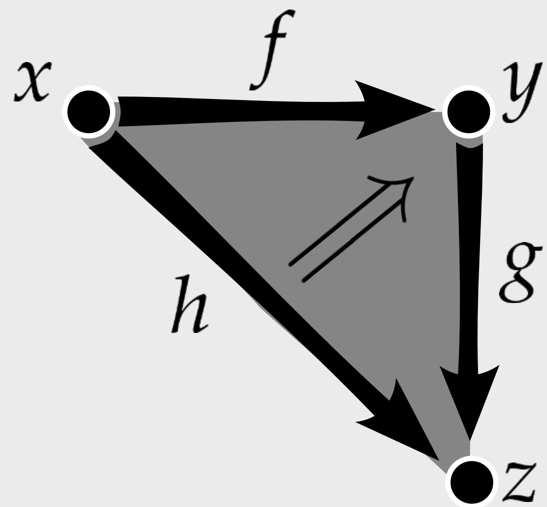


But first: what is the classifying space of a *poset-enriched* category?

Nervous breakdown

Every poset-enriched category $\mathbf{C}$ automatically produces a simplicial set $N\mathbf{C}$

Objects become vertices,
1-Morphisms are edges,
k-simplices span (k+1) morphisms that look like this:



Less pictorially: a k-simplex in $N\mathbf{C}$ consists of objects $x_0, \ldots, x_k$ along with morphisms $f_{ij} : x_i \to x_j, i \leq j$ so that $f_{im} \implies f_{ij} \circ f_{jm}$

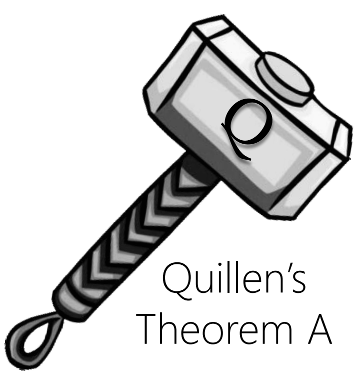
Poset-enriched functors yield simplicial maps, while (**lax** or **oplax**) natural transformations carry homotopies between them

Fiber optics

Given a functor $\mathbf{F} : \mathbf{C} \to \mathbf{D}$, its **fiber** over an object $d \in \mathbf{D}_0$ is a category $\mathbf{F}/\!/d$ defined as follows:

$$\begin{tikzcd} Fc \arrow[rr, "Ff"] \arrow[dr, "g"'] & & Fc' \arrow[dl, "g'"] \\ & d \arrow[u, phantom, "\Leftarrow"] & \end{tikzcd}$$

(This also goes by many other names: *over, comma, slice, arrow…*)



If $N(\mathbf{F}/\!/d)$ is contractible for every object, then $\mathbf{F} : \mathbf{C} \to \mathbf{D}$ induces a homotopy-equivalence of classifying spaces

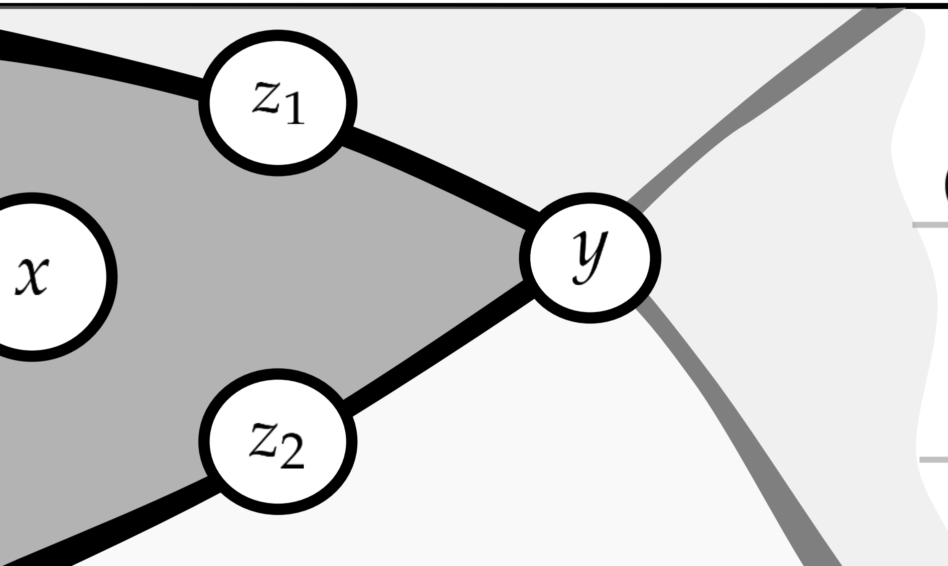
The entrance path category

Every regular CW complex X is homeomorphic to the classifying space of its face poset $\mathbf{Fac}(X)$, where $x > y$ records only that y is a face of x. *We need more*, and turn to a construction of R MacPherson.

The **entrance path category** of X fattens the face partial order while preserving its homotopy type. This is a poset-enriched category $\mathbf{Ent}(X)$ whose:

Objects are the cells, and
Morphisms are *strictly descending sequences* $(x > z_1 > \cdots > z_k > y)$

We compose these by concatenation; note that they are partially ordered (by refinement) and composition preserves this order.



The poset of entrance paths from x to y in this complex is:

$$(x > z_1 > y) \Leftarrow (x > y) \Rightarrow (x > z_2 > y)$$

The minimal path is *atomic,* and others factor "uniquely" into atoms:

$$(x > z_1 > y) = (x > z_1) \circ (z_1 > y)$$

$x \longrightarrow y$ will denote the atom

Morse pairings and entrance paths

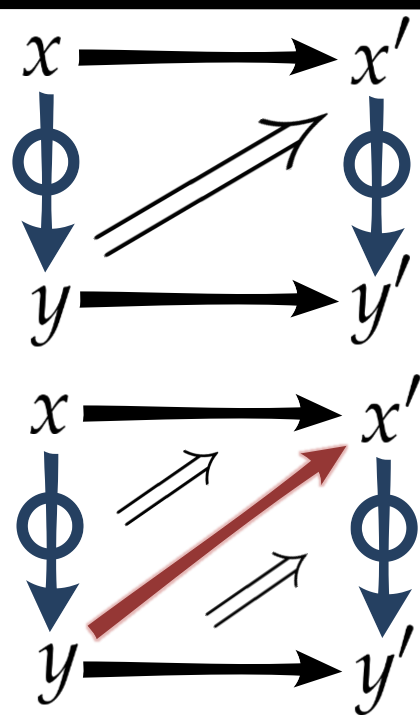
Any Morse pairing of cells in a regular CW complex X implicates atomic entrance paths $W = \{x_\bullet > y_\bullet\}$ in $\mathbf{Ent}(X)$ so that:

1. if $(x > y)$ is in W then x and y do not appear in any other pair of W,
2. the relation $(x > y) \blacktriangleright (x' > y')$ if $x > y'$ generates a partial order on W (i.e., the first pair precedes the second one in zigzag paths).

$$c > \ldots > y < x > \cdots > y' < x' > \cdots > c'$$

LIFT

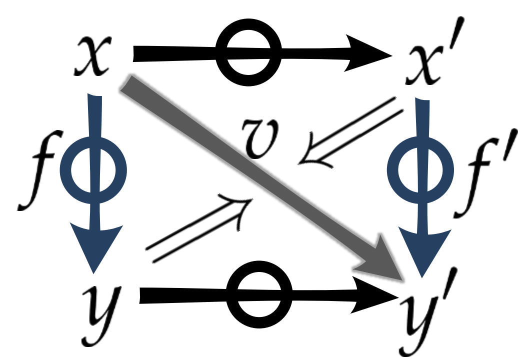
if



then

SWITCH

if



then $\quad y \xrightarrow{\exists q} x'$

and $\quad f \circ q \circ f' \Longrightarrow v$

Localization

We formally invert all entrance paths from W in order to produce a new zigzag-inspired poset-enriched category (1-cat version by W Dwyer & D Kan, 1980)

The **localization** of $\mathbf{Ent}(X)$ at W, written $\mathbf{Ent}_W(X)$, has the same objects, but morphisms are now equivalence classes of zigzags:

$$\begin{tikzcd} w \arrow[r] & y_0 & x_0 \arrow[l, "\circ" marking] \arrow[r] & \cdots \arrow[r] & y_k & x_k \arrow[l, "\circ" marking] \arrow[r] & z \end{tikzcd}$$

Only W-elements and identities can point backwards. These zigzags are partially ordered as you might expect:

$$\begin{tikzcd} w \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\Nearrow"] & y_0 \arrow[d, "\circ" marking] \arrow[dr, phantom, "\Nwarrow"] & x_0 \arrow[l, "\circ" marking] \arrow[r] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\Nearrow"] & \cdots \arrow[r] \arrow[dr, phantom, "\Nearrow"] & y_k \arrow[d, "\circ" marking] \arrow[dr, phantom, "\Nwarrow"] & x_k \arrow[l, "\circ" marking] \arrow[r] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\Nearrow"] & z \arrow[d, equal] \\ w \arrow[r] & y_0' & x_0' \arrow[l, "\circ" marking] \arrow[r] & \cdots \arrow[r] & y_k' & x_k' \arrow[l, "\circ" marking] \arrow[r] & z \end{tikzcd}$$

Only W-elements and identities can point downwards

The **localization functor** $\mathbf{L}_W : \mathbf{Ent}(X) \to \mathbf{Ent}_W(X)$

1. sends W-elements to isomorphisms, and
2. is a factor of *any other functor* which does the same...

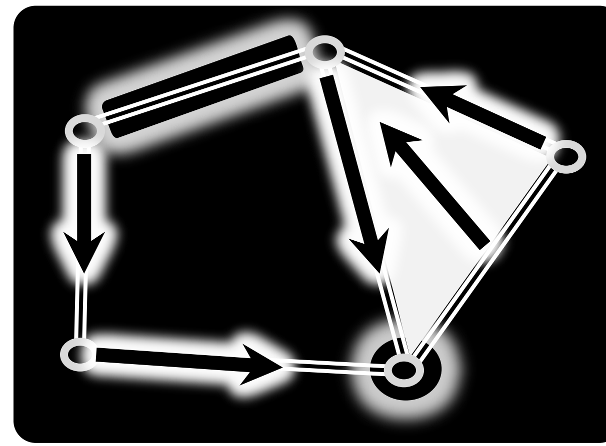
$$\begin{tikzcd} \mathbf{Ent}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \mathbf{C} \\ \mathbf{Ent}_W(X) \arrow[ur] & \end{tikzcd}$$

$\mathbf{L}_W$ is not mysterious: it just acts by inclusion

The discrete flow category

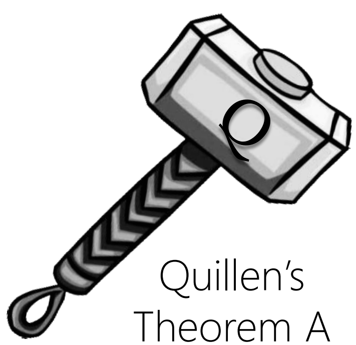
Let X be a regular CW complex equipped with a Morse pairing $W = \{x_\bullet > y_\bullet\}$

The **discrete flow** category $\mathbf{Flo}_W(X)$ is the *full subcategory* of the localization $\mathbf{Ent}_W(X)$ spanned by critical cells; that is,



- its objects are the critical cells (not included in W)
- its morphisms are (posets of equivalence classes of) zigzags as before: only W-elements and identities can point backwards, etc

Thm [N, 2015]: $B\mathbf{Flo}_W(X)$ is homotopy-equivalent to X.



Quillen's Theorem A

The localization functor $\mathbf{Ent}(X) \to \mathbf{Ent}_W(X)$ has contractible fibers, as does the inclusion $\mathbf{Flo}_W(X) \hookrightarrow \mathbf{Ent}_W(X)$. So, we have a co-span furnishing homotopy equivalences:

$$\mathbf{L}_W : \mathbf{Ent}(X) \stackrel{\simeq}{\to} \mathbf{Ent}_W(X) \stackrel{\simeq}{\hookleftarrow} \mathbf{Flo}_W(X) : \mathbf{J}_W$$

Outlines...

Localization fibers are contractible:

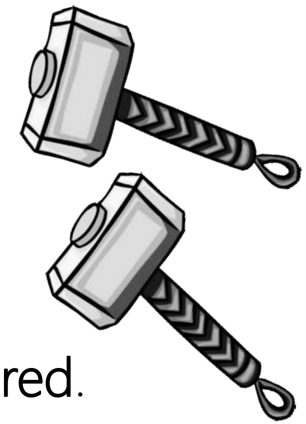
The fiber $\mathbf{L}_W /\!/ z$ over a cell z has objects which look like this:

$$w \longrightarrow y_0 \longleftarrow x_0 \longrightarrow \cdots \longrightarrow y_k \longleftarrow x_k \longrightarrow z$$

Assign to each object its “minimal z-augmented W-chain”, i.e.,

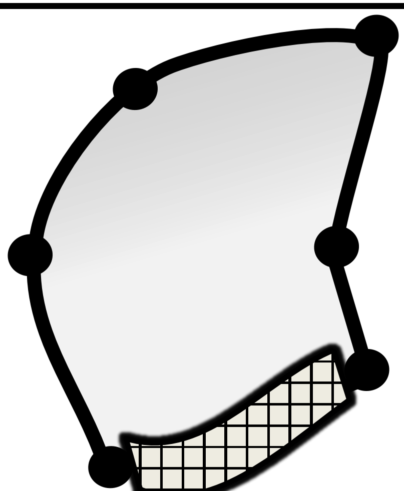
$$(x_0 > y_0) \blacktriangleright \cdots \blacktriangleright (x_k > y_k) \blacktriangleright (z)$$

(well-defined by Lift and order-preserving by Switch)



The assignment gives a new functor $\mathbf{N}_z$ from $\mathbf{L}_W /\!/ z$ to this poset, which contains (z) as a minimal element

Finally, show that $\mathbf{N}_z$ has contractible fibers: proof by Quillen-squared.



Inclusion fibers are also contractible:

Need *finiteness* to preclude infinite descending W-chains: i.e., every $(x_0 > y_0) \blacktriangleright (x_1 > y_1) \blacktriangleright \cdots$ must eventually stabilize

Need $\partial x \setminus y$ to be contractible for every pair $(x > y)$

Proceed by induction on $\blacktriangleright$

More generally, …

Call a 2-category $\mathbf{E}$ **cellular** if it is loop-free, and if every non-empty hom-category has an atomic (initial, indecomposable) element

A **Morse system** on $\mathbf{E}$ is a collection W of atoms which satisfy four (familiar) axioms: *exhaustion*, *order*, *lifting* and *switching*

Theorem: The localization functor $\mathbf{E} \to \mathbf{E}_W$ induces a homotopy equivalence

Call this Morse system **mild** if there are no infinite descending ►-chains, and if for each $f : x \to y$ in W, the full subcategory of $\mathbf{E}$ spanned by all the non-$\{x, y\}$ objects which admit morphisms from x is contractible

Here the discrete flow category $\mathbf{F}_W$ is defined as the full subcategory of $\mathbf{E}_W$ spanned by all objects untouched by morphisms in W

Theorem: If W is mild, then $\mathbf{F}_W \hookrightarrow \mathbf{E}_W$ also induces a homotopy equivalence

Applications

Very general Morse theory

Works even when the Morse pairings are made across codim > 1 [R Freij 09]

Morse theory for constructible (co)sheaves, stacks,...

$$
\begin{tikzcd}
\mathbf{Ent}(X) \arrow[r, "\mathbf{L}_W"] \arrow[dr, "F"'] & \mathbf{Ent}_W(X) \arrow[d] & \mathbf{Flo}_W(X) \arrow[l] \arrow[dl] \\
& \mathbf{A} &
\end{tikzcd}
$$

Why do wedges of spheres appear so often in combinatorics?

13

I think it's because we have well-developed techniques with which to prove that this condition holds, and when those fail, people don't put that much effort into trying to describe the (more difficult) homotopy types. I'd be happy to hear that this is an unduly pessimistic view.

share cite edit flag

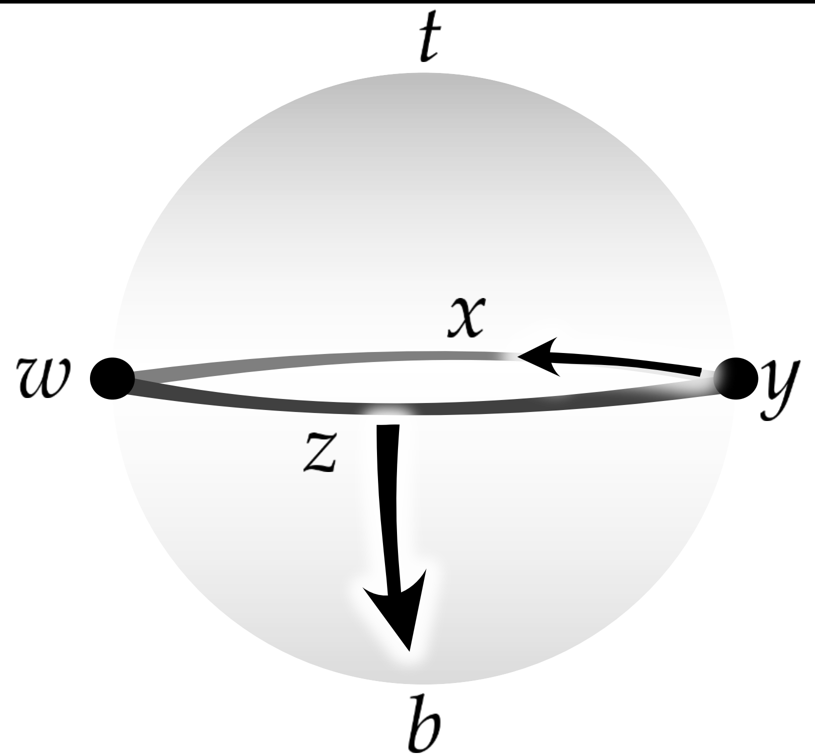
answered Mar 7 '10 at 13:33

Allen Knutson

17.9k ● 2 ● 35 ● 113

16 This is an unduly pessimistic view. – Steven Gubkin Mar 7 '10 at 15:18

Example

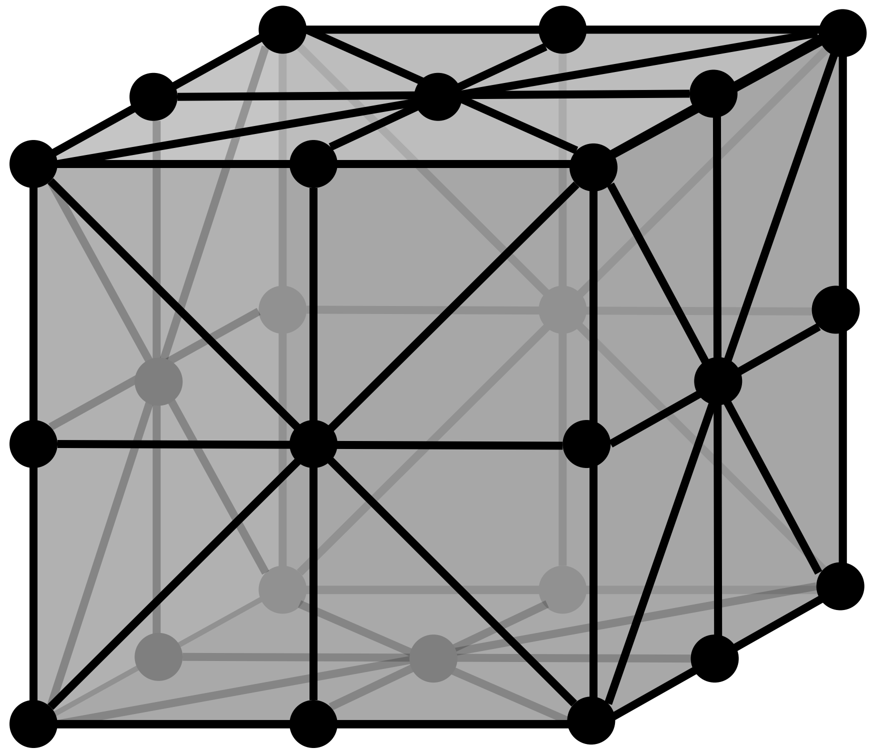


$$W = \{(b > z) \blacktriangleright (x > y)\}$$

The flow category has only two objects: the critical cells. So, its homotopy type depends entirely on the poset of zigzags from t to w

$$\begin{tikzcd}
(t > z > w) & (t > w) \arrow[l, Rightarrow] \arrow[r, Rightarrow] & (t > x > w) \\
(t > z < b > w) \arrow[u, Rightarrow] \arrow[d, Rightarrow] & & (t > y < x > w) \arrow[u, Rightarrow] \arrow[d, Rightarrow] \\
(t > z < b > x > w) & (t > z < b > y < x > w) \arrow[l, Rightarrow] \arrow[r, Rightarrow] & (t > z > y < x > w)
\end{tikzcd}$$

More on spheres



Theorem: The only non-empty hom-poset in the flow category of any perfect Morse matching on the minimal n-sphere is the Weyl chamber decomposition associated to the B_n root system

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