Relative Homological Algebra Via Truncations

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General framework

- In a previous talk we where shown "A path to the Grail."
- Here I will invite you to a cup of tea.
- But brewing good tea has its subtleties.

Framework

- We fix an abelian category $\mathcal{A}$ e.g. the category of R-modules for some ring R.
- We want to do homological algebra.

Framework

Basic tools to do homological algebra:

1. Class of short exact sequences.
2. Class of projective objects. Any object is covered by a projective.
3. Class of injective objects. Any object is embedded in one.

Can alter any of the three mutually related classes to do "relative homological algebra". Inspired by "cellularity" methods we will alter the meaning of "beeing an injective object".

Examples of relative settings

1. Restricting the s.e.s. to those that remain exact after tensoring by an objects leads to purity theory
2. Given a ring homomorphism $R \to S$ and short exact sequences of S-modules that are split as s.e.s. of R-modules (Hochschild classical framwork).
3. We could try to concentrate on resolutions of R-modules that only involve one prime ideal $p \subset R$. For instance by restricting the injectives to be direct sums of copies of $E(p)$.

Injective objects

- They behave well with respect to monomorphisms

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "\forall"'] & N \arrow[dl, dotted, "\exists"] \\ I & \end{tikzcd}$$

  Equivalently, for any monomorphism $f : M \hookrightarrow N$ and any injective object I, $Hom(f, I)$ is an epimorphism.
- Any module has a monomorphism into an injective
  $M \hookrightarrow I_M$

Fix a class $\mathcal{I}$ of objects in $\mathcal{A}$, objects you want to be your injectives. Without loss of generality, assume $\mathcal{I}$ is closed under products and retracts.

Definition

A map $f : M \to N$ is an $\mathcal{I}$-monomorphism, if and only if for any $I \in \mathcal{I}$, $Hom(f, I)$ is a surjection of abelian groups.

$$\begin{tikzcd} M \arrow[r, tail] \arrow[d, "\forall"'] & N \arrow[dl, dotted, "\exists"] \\ I & \end{tikzcd}$$

Definition (Enough relative injectives)

Any object M admits a $\mathcal{I}$-monomorphism into a relative injective: $M \rightarrowtail N$

Definition

A class of objects $\mathcal{I}$ in $\mathcal{A}$, closed under products and retracts, is an injective class if and only if every object embeds by an $\mathcal{I}$-monomorphism into an object in $\mathcal{I}$.

These are the objects with which we want to do homological algebra.

Definition

Denote by $Ch(\mathcal{A})$ the category of *homological* complexes: differentials lower degree by one.

$$X_\bullet : \cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow X_{n-2} \longrightarrow \cdots$$

Definition

A map of chain complexes $f : X_\bullet \to Y_\bullet$ is an $\mathcal{I}$-weak equivalence if and only if for any object $I \in \mathcal{I}$, the map of chain complexes of abelian groups:

$$Hom(f, I) : Hom(Y_\bullet, I) \to Hom(X_\bullet, I)$$

is a quasi-isomorphism.

We can rephrase our main objective: construct in an effective way the category:

$$D(\mathcal{A},\mathcal{I}) := Ch(\mathcal{A})[\mathcal{W}_I]^{-1}$$

- Effective: we want a "constructive way" to resolve unbounded complexes.
- Because we have enough relative injectives we know how to resolve objects.
- If you do not care bout set theroetical problems, this can be done formally, and get a large triangulated category.

Part of the solution

Fix $n \in \mathbb{Z}$ and let $Ch_{n\geq}(\mathcal{A})$ denote the full subcategory of "left bounded" complexes $X_k = 0$ for $k > n$

$$X_\bullet : \cdots \longrightarrow 0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow X_{n-2} \longrightarrow \cdots$$

There is a Quillen model structure on $Ch_{n\geq}(\mathcal{A})$

1. Weak equivalences: chain maps for which $Hom(f, I)$ is a quasi-isomorphism $\forall I \in \mathcal{I}$.
2. Cofibrations: $\mathcal{I}$-monorphisms in degrees $k < n$
3. Fibrations: Degree-wise split epimorphisms with $\mathcal{I}$-injective kernel.

Resolving left bounded complexes

Using this Quillen model structure, you can construct both

$$D_n(\mathcal{A};\mathcal{I}) = Ch(\mathcal{A})_{n\geq}[\mathcal{W}_{\mathcal{I}}]^{-1}$$

$$D_-(\mathcal{A};\mathcal{I}) = \bigcup_{n\in\mathbb{Z}} D_n(\mathcal{A};\mathcal{I})$$

Classification of Injective classes

Let $\mathcal{A} = R$-mod for a commutative Noetherian ring R.

- Have minimal injective resolutions of objects
- Every injective decomposes $I = \oplus_i E(R/p_i)$, for $p \in \mathrm{Spec}\ R$.
- Maps between ijectives are understood:

Lemma

Given $p, q \in Spec\ R$

$$Hom(E(R/p), E(R/q)) \neq 0 \Leftrightarrow p \subset q$$

- You can show that injective classes made of usual injectives are classified by the indecomposables they contain.

Theorem (CSP)

There is a bijection

$$\begin{array}{rcl} \begin{array}{c}\textit{Injective classes}\\ \textit{of injectives}\end{array} & \leftrightarrow & \begin{array}{c}\textit{Generization closed}\\ \textit{subsets of Spec } R\end{array} \\ \mathcal{I} & \mapsto & \{p \mid E(R/p) \in \mathcal{I}\} \\ \langle E(R/p) \,|\, p \in \mathcal{O}\rangle & \leftarrow & \mathcal{O} \end{array}$$

Generization closed = closed under taking primes inside your given prime.

Let $\mathcal{O} \subset \operatorname{Spec} R$ correspond to your favourite injective class.

- In this case relative monomorphisms are easy to recognize.
- An R-module map $f : M \to N$ is an $E(\mathcal{O})$-monomorphism if and only if $F_p : M_p \to N_p$ is a monomorphism for each $p \in \mathcal{O}$.

Resolutions of objects

Let $\mathcal{O} \subset \text{Spec } R$ correspond to your favourite injective class. Fix an injective resolution,

$$M \to I_0 \to I_{-1} \cdots$$

Each I_k decomposes as a direct sum of $E(R/p)$, p prime.

Lemma

Given $p, q \in \mathit{Spec}\ R$

$$\mathit{Hom}(E(R/p), E(R/q)) \neq 0 \Leftrightarrow p \subset q$$

ensures that keeping the summands not in your generization subset is a subcomplex.

$$M \to I_*/I_{*(p \notin \mathcal{O})}$$

is a relative resolution.

Resolving unbounded complexes

Idea to resolve an unbounded complex:

1. Approximate your unbounded complex by cutting it further and further to the left into pieces you control and approximate your complex.
2. Resolve each of your approximations.
3. Glue back the approximations.
4. Et voilà. (hopefully)

$$
\begin{tikzcd}
\cdots \arrow[r] & X_2 \arrow[r] & X_1 \arrow[r] & X_0 \arrow[r] & X_{-1} \arrow[r] & X_{-2} \arrow[r] & \cdots \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & X_2/X_3 \arrow[r, "d_2"] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & X_{-1} \arrow[r] \arrow[d] & X_{2,-2} \arrow[r] \arrow[d] & \cdots \\
 & 0 \arrow[r] & X_1/X_2 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & X_{-1} \arrow[r] \arrow[d] & X_{-2} \arrow[r] \arrow[d] & \cdots \\
 & & 0 \arrow[r] & X_0/X_1 \arrow[r] & X_{-1} \arrow[r] & X_{-2} \arrow[r] & \cdots
\end{tikzcd}
$$

The process of truncating further and further to the left and gluing back by taking the vertical inverse limit gives a pair of adjoint functors:

$$\mathsf{tow} : \mathsf{Ch}(\mathcal{A}) \rightleftarrows \mathsf{Tow}(\mathcal{A}, \mathcal{I}) : \lim$$

where $\mathit{Tow}(\mathcal{A}, \mathcal{I})$ is the category of towers of left bounded complexes:

- Each horizontal layer in the tower lands in a Quillen model category, with weak equivalences the $\mathcal{I}$-weak equivalences,
- all these categories are connected by Quillen pairs.

$$Tow(\mathcal{A}, \mathcal{I}): \quad \cdots \mathsf{Ch}_{2\geq}(\mathcal{A}) \leftrightarrow \mathsf{Ch}_{1\geq}(\mathcal{A}) \leftrightarrow \mathsf{Ch}_{0\geq}(\mathcal{A})$$

- At the end $Tow(\mathcal{A}, \mathcal{I})$ has a nice model category structure.
  - Weak equivalences and cofibrations are defined levelwise.
  - A tower is fibrant if the bottom layer is a fibrant complex (terms belong to $\mathcal{I}$), and each projection between layers isa degree-wise split epimorphism with kernel in $\mathcal{I}$.

Proposition

The pair of adjoint functors

$$tow : Ch(\mathcal{A}) \rightleftarrows \; Tow(\mathcal{A}, \mathcal{I}) : \; lim$$

form a right Quillen pair for $Ch(\mathcal{A})$ *with* $\mathcal{I}$*-weak equivalences as weak equivalences.*

- The right Quillen pair for the category with weak equivalences $(\mathcal{C}, \mathcal{W})$ $l : \mathcal{C} \rightleftarrows \mathcal{M} : r$ is a model approximation if it satisfies:
    - If $lA \to X$ is a weak equivalence in $\mathcal{M}$ and X is fibrant, then its adjoint $A \to rX$ is a weak equivalence in $\mathcal{C}$.
- In our case this means that the comparison map between a complex and the limit of the resolutions of its truncations should be an $\mathcal{I}$-weak equivalence.
- Then the essential image of $\mathcal{C}$ into $Ho(\mathcal{M})$ is a model for $\mathcal{C}[\mathcal{W}]^{-1}$ ([C-S]).

Try to resolve the complex with trivial differentials:

$$\cdots R \to R \to R \to 0 \to \cdots$$

Let R be Nagata's "bad noetherian ring": a Noetherian ring with a unique maximal ideal of height d for each integer $d > 0$. Choose the injective class corresponding to the generization subset of Spec R which is the complement of each of these maximal ideals. Then the above complex is not well approximated:

- The cohomology of the inverse limit of the resolutions is $\Pi_i E(R/\mathfrak{m}_i)$ in degree 0.
- The module $\Pi_i E(R/\mathfrak{m}_i)$ is not $\mathcal{I}$-trivial. Analogous to the fact that:

  $$\forall p \text{ prime} \quad \mathbb{Z}/p \otimes \mathbb{Q} = 0$$

  but

  $$(\Pi_p \quad \mathbb{Z}/p) \otimes \mathbb{Q} \neq 0$$

The "error" behaves however relatively tamely:

Theorem (CSP)

Let $f: \ tow(X) \rightarrow Y_\bullet$ be a weak equivalence in $Tow(\mathcal{A},\mathcal{I})$ and $g : X \rightarrow \ lim(Y_\bullet)$ be its adjoint. Then, for any $W \in \mathcal{I}$, $Hom(g, W)$ induces a split epimorphism on homology.

Theorem (CSP)

Let R be a Noetherian ring of finite Krull dimension. Then the right Quillen pair

$$tow : Ch(\mathcal{A}) \rightleftarrows \; Tow(\mathcal{A}, \mathcal{I}) : \; lim$$

is a right model approximation for any injective class of injectives.

The key problem is that in the (large) category $D(\mathcal{A}, \mathcal{I})$

$$X \not\cong \mathrm{holim}\,\tau_n X$$

this is due for instance, because countable products deviate from exactness in arbitrarily large degrees: derived functors of products are not trivial in arbitrarily large degrees.

Roos axiom AB4* $- n$

Definition

Let $\mathcal{A}$ be an abelian category, $\mathcal{I}$ an injective class and $n \geq 0$ an integer. We say that the category $\mathcal{A}$ satisfies axiom AB4*-$\mathcal{I}$-n if and only if, for any countable family of objects $(A_j)_{j \in J}$ and any choice of relative resolutions $A_j \to \mathcal{I}(A_j)_*$, the product complex $\Pi_{j \in J} \mathcal{I}(A_j)_*$ has vanishing $\mathcal{I}$-homology in degrees $k < -n$.

Theorem

If $\mathcal{A}$ satisfies axiom $AB4^*$-$n - \mathcal{I}$, then the right Quillen pair

$$tow : Ch(\mathcal{A}) \rightleftarrows \; Tow(\mathcal{A}, \mathcal{I}) : \; lim$$

is a right model approximation.

1. R-mod for R of finite injective dimension and any injective class of injectives.
2. A noetherian ring R and the injective class determined by an ideal of height n.
3. R-mod with the class of pure injective modules. If $|R| < \aleph_t$ then the pure injective dimension is $< t + 1$ (Kielpinski-Simson). In particular this category satisfies AB4*-$(t+2)$-(pure inj.)