# Categorical suspension and stable Postnikov data

(or "Towards modeling homotopy $n$-types of spectra II")

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# Towards modeling homotopy n-types of spectra

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# Motivation I: Picard groups

- Classical example: $R$ commutative ring. The set of isomorphism classes of invertible $R$-modules together with the tensor product forms a group $\mathrm{Pic}(R)$, the *Picard group of $R$*.
- More generally: $(C, \oplus, e)$ symmetric monoidal category. The set of isomorphism classes of invertible objects in $C$ together with the monoidal product forms a group $\mathrm{Pic}(C)$, the *Picard group of $C$*.

$$\textbf{SymMonCat} \to \textbf{PicardCat} \to \textbf{Top}$$

$$C \overset{\text{take invertible cells}}{\longmapsto} \mathrm{Core}(C) \longmapsto B\,\mathrm{Core}\,C$$

$$\pi_0(B\,\mathrm{Core}\,C) = Pic(C)$$
$$\pi_1(B\,\mathrm{Core}\,C) = Aut(e)$$

- $B\,\mathrm{Core}(C)$ is an infinite loop space: $\Omega^\infty K(\mathrm{Core}(C))$

# Motivation I: Picard groups

In addition to $Pic(R)$ and $Aut(R) = R^\times$, we are interested in the *Brauer group* $Br(R)$ of $R$.

- Have symmetric monoidal bicategory $\mathbf{Alg}(R)$:

| | | invertible |
|---|---|---|
| 0-cells | $R$-algebras $A$ | *Azumaya algebra* |
| 1-cells | $A$-$B$-bimodules $M\colon A \to B$ | *Morita equivalence* |
| 2-cells | bimodule hom. $f\colon M \to N$ | isomorphism |

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$$Br(R) := \{\text{Azumaya algebras}\}/\text{Morita equivalence}$$

$$Pic(R) = \{\text{Morita equivalences } R \to R\}/\text{isomorphism}$$

$$R^\times = \{\text{automorphisms of the } R\text{-}R\text{-bimodule } R\}$$

# Motivation I: Picard groups

- $(\mathcal{D}, \oplus, e)$ symmetric monoidal bicategory

$$\mathcal{D} \overset{\text{take invertible cells}}{\longmapsto} \mathrm{Core}(\mathcal{D}) \longmapsto B\,\mathrm{Core}(\mathcal{D})$$

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$$\pi_0(B\,\mathrm{Core}(\mathcal{D})) \cong \{\text{invertible objects}\}/\text{invertible 1-cells}$$
$$\pi_1(B\,\mathrm{Core}(\mathcal{D})) \cong \{\text{invertible 1-cells } e \to e\}/\text{isomorphisms}$$
$$\pi_2(B\,\mathrm{Core}(\mathcal{D})) \cong \{\text{automorphisms of the 1-cell } \mathrm{id}_e : e \to e\}$$

- Want: $B\,\mathrm{Core}(\mathcal{D})$ is an infinite loop space: $\Omega^\infty K(\mathrm{Core}(D))$
- Need: $K$

# Motivation II: Algebraic/categorical models for homotopy types

### Stable Homotopy Hypothesis

*The category of Picard n-categories with the categorical equivalences and the category of stable n-types have equivalent homotopy categories.*

- Holds for $n = 0$: Picard 0-categories are abelian groups
- Unstable analogue: Grothendieck's Homotopy Hypothesis, many results (Whitehead, MacLane-Whitehead, Loday, Brown, Moerdijk-Svensson, . . . )

# What we know and what we would like to know

| $n = 1$ | $n = 2$ |
| --- | --- |
| $K\colon \mathbf{SymMonCat} \to \mathbf{Sp}_{[0,\infty]}$ induces an equivalence on homotopy categories (Thomason) | $K\colon \mathbf{SymMonBiCat} \to \mathbf{Sp}_{[0,\infty]}$ induces an equivalence on homotopy categories (GJO) |
| Stable Homotopy Hypothesis holds (folklore, several) | Stable Homotopy Hypothesis holds (GJO, forthcoming work) |
| Every Picard category is equivalent to a skeletal and strict one. | ? |
| In particular, have nice model for the 1-truncation of the sphere spectrum $S$. | ? |
| Express $k$-invariant categorically. | ? |

# Strict symmetric monoidal $n$-categories

Actually, the $K$-theory functors are defined for strict symmetric monoidal categories and 2-categories.

$n = 1$

Every symmetric monoidal category is equvialent to a strict one. (MacLane)

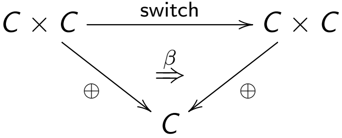
$n = 2$

Every symmetric monoidal bicategory is equivalent to a strict symmetric monoidal 2-category. (Schommer-Pries, GO)

# Strict symmetric monoidal $n$-categories

$n = 1$

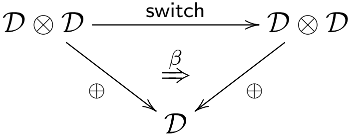
A *strict symmetric monoidal category* is a monoid $(C, \oplus\colon C \times C \to C, e)$ in $(\mathbf{Cat}, \times, *)$ together with a natural isomorphism




satisfying some axioms.

$n = 2$

A *strict symmetric monoidal* 2-*category* is a monoid $(\mathcal{D}, \oplus\colon \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}, e)$ in $(\mathbf{2Cat}, \text{Gray } \otimes, *)$ together with a 2-natural isomorphism




satisfying some axioms.

# Strict Picard $n$-categories

$n = 1$

$(C, \oplus, e)$ is a *strict Picard category* if every cell of $C$ is invertible

$n = 2$

$(\mathcal{D}, \oplus, e)$ is a *strict Picard 2-category* if every cell of $\mathcal{D}$ is invertible

# Postnikov tower of a connective spectrum $X$

$$
\begin{array}{ccccc}
 & & \vdots & & \\
\Sigma^2 H(\pi_2 X) & \xrightarrow{i_2} & X_2 & \xrightarrow{k_2} & \Sigma^4 H(\pi_3 X) \\
 & & \downarrow & & \\
\Sigma H(\pi_1 X) & \xrightarrow{i_1} & X_1 & \xrightarrow{k_1} & \Sigma^3 H(\pi_2 X) \\
 & & \downarrow & & \\
 & & X_0 = H(\pi_0 X) & \xrightarrow{k_0} & \Sigma^2 H(\pi_1 X)
\end{array}
$$

# $k_0$ algebraically

## Facts (Eilenberg-MacLane, '54)

- There is a natural isomorphism

$$[HA, \Sigma^2 HB] \cong \mathbf{Ab}(A \otimes \mathbb{Z}/2, B)$$

for any abelian groups $A$, $B$.

- Under this identification:

$$[H(\pi_0 X), \Sigma^2 H(\pi_1 X)] \cong \mathbf{Ab}((\pi_0 X) \otimes \mathbb{Z}/2, \pi_1 X)$$

$$k_0 \mapsto (\text{precomposition with the Hopf element } \eta \colon \Sigma S \to S)$$

# Example: $X = S$ the sphere spectrum

$$
\begin{array}{ccccc}
\Sigma^2 H(\mathbb{Z}/2) & \xrightarrow{i_2} & X_2 & & \\
 & & \downarrow & & \\
\Sigma H(\mathbb{Z}/2) & \xrightarrow{i_1} & X_1 & \xrightarrow{k_1} & \Sigma^3 H(\mathbb{Z}/2) \\
 & & \downarrow & & \\
 & & X_0 = H\mathbb{Z} & \xrightarrow{k_0} & \Sigma^2 H(\mathbb{Z}/2)
\end{array}
$$

- $\pi_0(S) \otimes \mathbb{Z}/2 \to \pi_1(S)$ is an isomorphism, $\mathrm{id}_S \otimes 1 \mapsto \eta$
- $k_1 i_1$ corresponds to $Sq^2 \in (H\mathbb{Z}/2)^2(H\mathbb{Z}/2)$

# Main result I: triviality of some Postnikov data

### Definition

A strict Picard 2-category is called *skeletal* if it satisfies the following condition: if there exists an invertible 1-cell between two objects $x$ and $y$, then $x = y$.

### Theorem (GJOS)

*Let $\mathcal{D}$ be a skeletal, strict Picard 2-category. If $k_0 \colon \pi_0 K\mathcal{D} \otimes \mathbb{Z}/2 \to \pi_1 K\mathcal{D}$ is surjective, then $k_1 i_1$ is trivial in $[\Sigma H(\pi_1 K\mathcal{D}), \Sigma^3 H(\pi_2 K\mathcal{D})]$.*

### Corollary

*There is no skeletal, strict Picard 2-category whose $K$-theory spectrum realizes the 2-truncation of the sphere spectrum.*

# Stable Postnikov data of strict Picard $n$-categories

| | |
|---|---|
| $n = 1$ | $n = 2$ |
| $C$ strict Picard category | $\mathcal{D}$ strict Picard 2-category |
| $BC = \Omega^\infty KC$ | $B\mathcal{D} = \Omega^\infty K\mathcal{D}$ |
| $\pi_0(KC) = \mathrm{ob}C/1\text{-cells}$ | $\pi_0(K\mathcal{D}) = \mathrm{ob}\mathcal{D}/1\text{-cells}$ |
| $\pi_1(KC) = C(e, e)$ | $\pi_1(K\mathcal{D}) = \text{ob}\mathcal{D}(e, e)/2\text{-cells}$ |
| | $\pi_2(K\mathcal{D}) = \mathcal{D}(e, e)(\mathrm{id}_e, \mathrm{id}_e)$ |
| $k_0 \colon \pi_0(KC) \otimes \mathbb{Z}/2 \to \pi_1(KC)$ | $k_0 = ?$, $k_1 = ?$ or at least $k_1 i_1 = ?$ |

$$[x] \otimes 1 \mapsto (e \cong xxx^*x^* \xrightarrow{\beta_{x,x}x^*x^*} xxx^*x^* \cong e)$$

# An adjunction modelling the 1-truncation

## Proposition

*The functors*

$$\begin{aligned}(-)_1\colon \textbf{StrictSymMon2Cat} &\rightleftarrows \textbf{StrictSymMonCat}\colon d\\ (obC, C(x,y), \textit{identity 2-cells}) &\leftarrow C\\ \mathcal{D} &\mapsto (ob\mathcal{D}, \pi_0\mathcal{D}(x,y))\end{aligned}$$

*form an adjunction. If $\mathcal{D}$ is a strict Picard 2-category, then*

$$K(\mathcal{D}) \to K(d(\mathcal{D}_1))$$

*is the 1-truncation of $K(\mathcal{D})$.*

# Bottom stable Postnikov invariant

It is straightforward to check that $K(C) \simeq K(dC)$ for any strict symmetric monoidal category $C$.

## Corollary

*Let $\mathcal{D}$ be a strict Picard 2-category with unit $e$ and symmetry $\beta$. The bottom stable Postnikov invariant*

$$k_0\colon H(\pi_0 K\mathcal{D}) \to \Sigma^2 H(\pi_1 K\mathcal{D})$$

*is modelled by the map $\pi_0(K\mathcal{D}) \otimes \mathbb{Z}/2 \to \pi_1(K\mathcal{D})$*

$$[x] \otimes 1 \mapsto [e \simeq xxx^*x^* \xrightarrow{\beta_{x,x}x^*x^*} xxx^*x^* \simeq e],$$

*where $x$ is an object of $\mathcal{D}$ and $x^*$ denotes an inverse of $x$.*

# An adjunction modelling the 0-connected cover

### Proposition

*The functors*

$$\Sigma \colon \textbf{StrictSymMonCat} \rightleftarrows \textbf{StrictSymMon2Cat} \colon \Omega$$

$$C \mapsto (*, obC, morC)$$

$$\mathcal{D}(e, e) \leftarrow\!\shortmid \mathcal{D}$$

*form an adjunction. If $\mathcal{D}$ is a strict Picard 2-category, then*

$$K(\Sigma\Omega\mathcal{D}) \to K(\mathcal{D})$$

*is the 0-connected cover of $K(\mathcal{D})$.*

# Main result II: categorical suspension

**Theorem (GJOS)**

*For any strict symmetric monoidal category $C$, the spectra $\Sigma K(C)$ and $K(\Sigma C)$ are stably equivalent.*

# First Postnikov layer $k_1 i_1$

## Corollary

*Let $(\mathcal{D}, \oplus, e)$ be a strict Picard 2-category with Gray structure 2-cells $\Sigma_{f,g}$ under $\oplus$. The composite*

$$k_1 i_1 \colon \Sigma H(\pi_1 K\mathcal{D}) \to \Sigma^3 H(\pi_2 K\mathcal{D})$$

*is modelled by the map $\pi_1(K\mathcal{D}) \otimes \mathbb{Z}/2 \to \pi_2(K\mathcal{D})$*

$$[f] \otimes 1 \mapsto [\mathrm{id}_e \cong f \circ f \circ f^* \circ f^* \xrightarrow{\Sigma_{f,f} f^* \circ f^*} f \circ f \circ f^* \circ f^* \cong \mathrm{id}_e],$$

*where $f \colon e \to e$ is a 1-cell of $\mathcal{D}$ and $f^*$ denotes an inverse of $f$.*

# Back to main result I

## Theorem (GJOS)

*Let $\mathcal{D}$ be a skeletal, strict Picard 2-category. If $k_0\colon \pi_0 K\mathcal{D} \otimes \mathbb{Z}/2 \to \pi_1 K\mathcal{D}$ is surjective, then $k_1 i_1$ is trivial.*

- $k_1 i_1\colon \pi_1(K\mathcal{D}) \otimes \mathbb{Z}/2 \to \pi_2(K\mathcal{D})$
  $[f] \otimes 1 \mapsto [\mathrm{id}_e \cong f \circ f \circ f^* \circ f^* \xrightarrow{\Sigma_{f,f} f^* \circ f^*} f \circ f \circ f^* \circ f^* \cong \mathrm{id}_e]$
- To show: $\Sigma_{f,f}$ is an identity 2-cell
- Use of assumptions:

  $$k_0\colon [x] \otimes 1 \mapsto [e \simeq xxx^*x^* \xrightarrow{\beta_{x,x}x^*x^*} xxx^*x^* \simeq e]$$

  surjective and $\mathcal{D}$ skeletal
- Reduction to the case $f = \beta_{x,x}x^*x^*$.