Rational global homotopy theory

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Report on my ongoing PhD project.

Main Goal

Explain how geometric fixed points can be used to give an algebraic model for the rational global homotopy category.

Note:
We only deal with finite groups, so this is actually fin-global homotopy theory.

Quick reminder about global homotopy theory

Let $X \in \mathrm{Sp}^{\mathrm{O}}$ be an orthogonal spectrum.

- G: finite group $\rightsquigarrow$ can regard X as a G-orthogonal spectrum $X\langle G\rangle \in G\text{-}\mathrm{Sp}^{\mathrm{O}}$ with trivial G-action.
  By evaluating at orthogonal G-representations V we obtain non-trivial G-spaces $X(V)$.
- This gives us the *equivariant homotopy groups* of X:

$$\pi_k^G X = \pi_k^G X\langle G\rangle = \mathrm{colim}_{n\in\mathbb{N}}[S^{k+n\cdot\rho_G}, X(n\cdot\rho_G)]_G,$$

  where $k \in \mathbb{Z}$ and $\rho_G = \mathbb{R}[G] =$ regular representation of G.
- A map $f : X \to Y$ is called a *global equivalence* if it induces isomorphisms on all equivariant homotopy groups

The global homotopy category

Definition

The *global homotopy category* is the localization

$$\mathcal{GH} = \mathrm{Sp}^{\mathrm{O}}[(\text{gl. equivalences})^{-1}].$$

- This is the homotopy category of a stable monoidal model category, in particular a tensor triangulated category.
- A set of compact generators is given by (suspension spectra of) *global classifying spaces* $B_{\mathrm{gl}}G$ which corepresent the equivariant homotopy groups:

$$\pi_k^G X = [\Sigma^k B_{\mathrm{gl}} G_+, X]_{\mathcal{GH}}.$$

- Global classifying spaces: The underlying K-homotopy type $(B_{\mathrm{gl}}G)\langle K\rangle \simeq B\mathcal{F}(K;G)$ is that of a classifying space for the family of *graph subgroups* of $K \times G$.

Global functors

Algebraic structure as G varies:

- For every homomorphisms of groups $\alpha : G \to K$ there is a *restriction map* (only depends on the conjugacy class)

$$\mathrm{res}_\alpha : \pi_*^K X \longrightarrow \pi_*^G X.$$

- For subgroup inclusions $H \leq G$ we also have *transfer maps*

$$\mathrm{tr}_H^G : \pi_*^H X \longrightarrow \pi_*^G X.$$

- Transfers commute with restrictions along surjections and there is a *double coset formula* for restrictions along subgroup inclusions.

Notation:

$$\mathcal{GF} = \text{the abelian category of } \textit{global functors}$$

Geometric fixed points

In equivariant stable homotopy theory there are also *geometric fixed point* homotopy groups:

$$\Phi_k^G X = \operatorname{colim}_{n \in \mathbb{N}} [S^{k+n}, X(n \cdot \rho_G)^G].$$

Have a comparison map, the *geometric fixed points map*:

$$\phi^G : \pi_*^G X \longrightarrow \Phi_*^G X,$$

Functoriality:

- There are still restrictions, but only along surjections

$$\alpha : G \twoheadrightarrow K \rightsquigarrow \operatorname{res}_\alpha : \Phi_*^K X \to \Phi_*^G X$$

- Only depends on the conjugacy class of α, compatible with composition

Geometric fixed points

Definition

Let Out be the category with

objects: finite groups

morphisms: conjugacy classes of epimorphisms

So the geometric fixed points are contravariant functors from the category Out:

$$\Phi_*^{(-)}X \in \mathrm{Out}^{\mathrm{op}}\text{-mod}$$

The geometric fixed points map commutes with restrictions and annihilates transfers from proper subgroups:

$$\phi^G(\mathrm{tr}_H^G(x)) = 0 \in \Phi_*^G X, \quad H \lneq G.$$

Time to rationalize

Question

How does the global homotopy category simplify if we look at it rationally ?

Definition

The *rational global homotopy category* is the localization

$$\mathcal{GH}_{\mathbb{Q}} = \mathrm{Sp}^{\mathrm{O}}[(\text{rational gl. equivalences})^{-1}].$$

Rational global equivalence:
A map inducing isomorphisms on all $\pi_*^G(\_) \otimes \mathbb{Q}$.

Here is some evidence that it's reasonable to look at geometric fixed points:

Proposition

There is an equivalence of abelian categories

$$\tau : \mathcal{GF}_{\mathbb{Q}} \xrightarrow{\simeq} \mathrm{mod}_{\mathbb{Q}}\text{-}\,\mathrm{Out}\,.$$

The functor τ is defined by dividing out proper transfers:

$$\tau(F)(G) = F(G)/\sum_{H \lneq G} \mathrm{im}(\mathrm{tr}_H^G), F \in \mathcal{GF}$$

Proposition

The geometric fixed points map factors over a rational isomorphism

$$\phi : \tau(\pi_* X) \xrightarrow{\cong_{\mathbb{Q}}} \Phi_* X$$

Strategy:
Use geometric fixed point spectra to produce a functor

$$\mathrm{Sp}^{\mathrm{O}} \longrightarrow \mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{Sp}^{\mathrm{O}}$$

which induces an equivalence

$$\mathcal{GH}_{\mathbb{Q}} \xrightarrow{\simeq} \mathbf{Ho}(\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{Sp}^{\mathrm{O}})_{\mathbb{Q}} \simeq \mathcal{D}(\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}}).$$

(Doesn't quite work that way)

Geometric fixed points spectra

There is a nice model $\Phi^G X \in \mathrm{Sp}^{\mathrm{O}}$ for the geometric fixed points spectrum of X:
At an inner product space V it is given by

$$(\Phi^G X)(V) = X(V \otimes \rho_G)^G.$$

Functoriality in G:

- Given a surjection $\alpha : G \twoheadrightarrow K$ summation over the fibers defines a G-equivariant isometry

$$\alpha^* \rho_K \hookrightarrow \rho_G, \quad g \mapsto \frac{1}{\sqrt{|\ker \alpha|}} \sum_{g \in \alpha^{-1}(k)} g$$

- This induces a map

$$X(V \otimes \rho_K)^K = X(V \otimes \alpha^* \rho_K)^G \to X(V \otimes \rho_G)^G$$

Geometric fixed points spectra

Putting these together we obtain a map of spectra

$$\alpha^* : \Phi^K X \to \Phi^G X$$

and we end up with

$$\Phi^{(\text{-})} X \in \mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Sp}^{\mathrm{O}}\,.$$

Here Epi is the category of finite groups and surjective homomorphisms.

Example

$$\Phi^K B_{\mathrm{gl}} G_+ \cong \bigvee_{\psi \in \mathrm{Rep}(K,G)} BC(\psi)_+$$

is a wedge of classifying spaces of centralizers indexed by conjugacy classes of homomorphisms.

Identifying the rational global homotopy category

We form the following composite:

$$\kappa : \mathrm{Sp}^{\mathrm{O}} \xrightarrow{\Phi} \mathrm{Epi}^{\mathrm{op}}\text{-}\,\mathrm{Sp}^{\mathrm{O}} \xrightarrow{c} \mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}} \xrightarrow{Lan} \mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}}$$

- c is defined by applying a *rational chain functor* $c : \mathrm{Sp}^{\mathrm{O}} \to \mathrm{Ch}_{\mathbb{Q}}$, $H_* cX \cong (\pi_* X)_{\mathbb{Q}}$ at each group G
- The left Kan extension just divides out by the conjugation actions:

$$Lan(C(\_))(G) = C(G)/\operatorname{Inn}(G), \quad C \in \mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}}$$

- Rationally this is exact and commutes with taking homology
- So if C arises as the geometric fixed points of a spectrum X this has no effect on homology

Identifying the rational global homotopy category

Upshot:

- We obtain a functor κ together with a natural isomorphism $H_*\kappa X \cong (\Phi_* X)_{\mathbb{Q}}$. In particular it is homotopical and descends to a functor on homotopy categories.
- $\kappa(B_{\mathrm{gl}} G_+) \simeq \mathbb{Q}[\mathrm{Rep}(\_, G)]$ (concentrated in degree 0) and these form a set of compact generators of $\mathcal{D}(\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}})$
- Calculational fact: Fully faithful on global classifying spaces

Theorem (W.)

Geometric fixed points induce a monoidal equivalence

$$\mathcal{GH}_{\mathbb{Q}} \xrightarrow{\simeq} \mathcal{D}(\mathrm{mod}_{\mathbb{Q}}\text{-}\,\mathrm{Out})$$

between the rational global homotopy category and the derived category of rational $\mathrm{Out}^{\mathrm{op}}$*-modules with the groupwise tensor product of chain complexes.*

Decompositions into Eilenberg-MacLane objects

- Classical case: Rational G-Mackey functors are projective $\implies$ rational G-spectra split as a wedges of Eilenberg-MacLane spectra.
- Globally this is no longer true. The abelian category $\mathcal{GF}_{\mathbb{Q}} \simeq \mathrm{mod}_{\mathbb{Q}}\text{-}\,\mathrm{Out}$ does not split.

Example (Non-projective)

Let $\underline{\mathbb{Q}} = \mathbb{Q}[\mathrm{Out}(\_, e)]$ be the constant $\mathrm{Out}^{\mathrm{op}}$-module (projective) and $\mathbb{Q}_e = \mathbb{Q}$ concentrated at the trivial group. Then the surjection

$$\underline{\mathbb{Q}} \twoheadrightarrow \mathbb{Q}_e$$

does not split.

The symmetric powers Sp^n of the sphere spectrum are naturally ocuring global homotopy types that are not Eilenberg-MacLane spectra (Markus Hausmann).

Ex: Global K-theory

- Global K-theory: Global homotopy type that restricts to equivariant K-theory for every finite group.
- Homotopy groups are given by the *representation ring* global functor $\mathbf{RU}$ in even degrees.

Theorem (W.)

For any rational global functor F the higher Ext*-groups*

$$\mathrm{Ext}^n_{\mathcal{GF}_\mathbb{Q}}(F, \mathbf{RU} \otimes \mathbb{Q}) = 0, \quad n \geq 2$$

vanish.

Corollary

The global K-theory spectrum KU splits as a wedge of global Eilenberg-MacLane spectra.

Real global homotopy theory

- One can also set up a *Real* (in the C_2-equivariant sense) version of global homotopy theory
- If we think of global homotopy types as encoding actions of all finite groups G, then Real global homotopy types encode twisted actions by all augmented groups $G \to C_2$
- Naturally occuring examples are Atiyah's KR-theory and Real Bordism MR.

Real global homotopy theory

Rationally there is again an equivalence

$$\mathcal{GHR}_{\mathbb{Q}} \simeq \mathcal{D}(\mathrm{Out}_R^{\mathrm{op}}\text{-mod}_{\mathbb{Q}})$$

Out_R: Real version of Out

- Objects: augmented groups $G \to C_2$
- Morphism: surjections over C_2 but with a additional grading which is twisted by conjugations