# COHOMOLOGY OF GROUPS : EXERCISES

Sheet 1, 4 April 2007

Prerequisites on simplicial sets (and the solution to some of the exercises below!) can be found in Waldhausen's notes “Algebraische Topologie”, chapter 9, available at `http://www.math.uni-bonn.de/people/schwede/Lehre/Lehre-index.html`.

**Definition.** Let $C$ be a small category, i.e., the class of objects of $C$ is a set. Define the *nerve* of $C$ to be the simplicial set $NC$ corepresented by $\Delta$ in the category of small categories and functors. More explicitly, recall that $\Delta$ is the category with set of objects $\{[n] \,|\, n \in \mathbb{Z},\ n \geqslant 0\}$. Here $[n]$ is the ordered set $\{0, 1, \dots, n\}$ viewed as a category (we have exactly one morphism $k \to \ell$ if $k \leqslant \ell$, and no morphism otherwise). The morphisms from $[n]$ to $[m]$ in $\Delta$ are the functors. The nerve of $C$ is given by the contravariant functor

$$\begin{aligned} NC : \Delta &\to \text{Sets} \\ [n] &\mapsto \text{Functors}([n], C)\,. \end{aligned}$$

The geometric realization $|NC|$ of $NC$ is called the *classifying space* of $C$, and is denoted $BC$.

**Exercise 1.1.** If $C$ is small category and $n \geqslant 0$ is an integer, give an explicit description of an $n$-simplex in $N_nC$. Using this description, express the face map $d_i : N_nC \to N_{n-1}C$ and the degeneracy map $s_i : N_nC \to N_{n+1}C$, for $0 \leqslant i \leqslant n$.

**Exercise 1.2.** Let $[n] \in \Delta$ be viewed as a small category, as mentioned above. Prove that $B[n]$ is homeomorphic to the standard $n$-simplex $\nabla^n \subset \mathbb{R}^{n+1}$.

**Exercise 1.3.** Let $C$ and $D$ be small categories, $F$ and $G$ be functors $C \to D$, and let $\eta : F \to G$ be a natural transformation. Show that $BF$ and $BG$ are freely homotopic as maps $BC \to BD$.

*Hint:* construct the homotopy $H : \nabla_1 \times BC \to BD$ between $BF$ and $BG$ as the realization of the nerve of a functor $[1] \times C \to D$.

**Exercise 1.4.** Let $G$ be a group and $X$ be a connected simplicial object in free $G$-sets, i.e. a contravariant functor $X : \Delta \to G{-}\text{Sets}$ such for all $n \geqslant 0$ the $G$-action on $X_n$ is free. Prove that the natural map $|X|/G \to |X/G|$ is a homeomorphism. Prove that $G$ acts properly on $|X|$, i.e., every $x \in |X|$ has a neighborhood $V$ such that $V \cap gV = \emptyset$ for all $g \in G - \{1\}$. Deduce that $|X| \to |X/G|$ is a covering with group of deck transformations isomorphic to $G$.