COHOMOLOGY OF GROUPS : EXERCISES

Sheet 2, 16 April 2007

Exercise 2.1. Let G be a group. Exhibit a contracting homotopy for the augmented bar resolution $F_G \to \mathbb{Z}$ and the augmented normalized bar resolution $\bar{F}_G \to \mathbb{Z}$, in the category of chain complexes of $\mathbb{Z}$-modules. Do such homotopies exist in the category of chain complexes of $\mathbb{Z}[G]$-modules ?

Exercise 2.2. Let $g_1, \dots, g_n$ be elements in a group G that pairwise commute. Let S_n be the group of permutations of a set with n elements. Show that

$$z(g_1, \dots, g_n) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)[g_{\sigma(1)}| \dots |g_{\sigma(n)}]$$

is a cycle in $\mathbb{Z} \otimes_{\mathbb{Z}[G]} F_G$.

Exercise 2.3. Let $p \geqslant 3$ be a prime, C_p be the cyclic group of order p generated by an element t, and $\mathbb{Z}$ be given the trivial C_p-module structure (i.e., t is the identity of $\mathbb{Z}$). Show that

$$H_n(C_p; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/p & \text{if } n \geqslant 1 \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

For $n \geqslant 1$ odd, find a representant of a generator of $H_n(C_p; \mathbb{Z})$ in $\mathbb{Z} \otimes_{\mathbb{Z}[G]} F_G$.

Exercise 2.4. Let G be a group and M a $\mathbb{Z}[G]$-module with G-action given by a homomorphism $\phi : G \to \operatorname{Aut}(M)$. Let C be the set of subgroups H of $M \rtimes_\phi G$ such that $H \cap M = \{1\}$ and $MH = M \rtimes_\phi G$. We define an equivalence relation $\sim$ on C by setting $H \sim H'$ if H and H' are conjugate subgroups of $M \rtimes_\phi G$. Construct an isomorphism of sets

$$H^1(G, M) \xrightarrow{\cong} C/\sim .$$

Hint : represent elements of $H^1(G, M)$ by cocycles $f : G \to M$ in the cocomplex $\hom_{\mathbb{Z}[G]}(F_G, M)$

Exercise 2.5. Describe $\operatorname{Aut}(\mathbb{Z}/n)$ for $n \geqslant 2$.

Exercise 2.6. Let $n \geqslant 3$ be an integer and let D_{2n} be the group of symmetries of a regular polygon with n sides in $\mathbb{R}^2$. It is called the *dihedral* group of order $2n$. Show that there is an isomorphism

$$D_{2n} \cong \mathbb{Z}/n \rtimes_\phi \mathbb{Z}/2$$

for $\phi : \mathbb{Z}/2 \to \operatorname{Aut}(\mathbb{Z}/n)$ sending the generator to multiplication by -1.

Exercise 2.7. Classify all extensions $1 \to \mathbb{Z}/4 \to E \to \mathbb{Z}/2 \to 1$ up to isomorphism.