COHOMOLOGY OF GROUPS : EXERCISES

Sheet 3, 26 April 2007

Exercise 3.1. Compute $H^2(\mathbb{Z}/2, \mathbb{Z}/4)$ for any action of $\mathbb{Z}/2$ on $\mathbb{Z}/4$. Is the result compatible with your solution of Exercise 2.7 ?

Exercise 3.2. Let (G, N, ϕ) be an abstract kernel, and C be the center of N with G-action induced by ϕ. Prove that the group $H^2(G, C)$ acts freely and transitively on the set of isomorphism classes of extentions $\mathcal{E}(G, N, \phi)$ (see the hints given in the lecture).

Exercise 3.3. Let G be a group different then $\mathbb{Z}/2$, and C be a G-module. Prove that any element α of $H^3(G, C)$ can be represented as the obstruction to realizing an abstract kernel (G, N, ϕ).

Hint: Let F be the (non-commutative) free group generated by the set

$$\{\ [x, y] \mid x, y \in G - \{1\}\ \},$$

and take $N = C \times F$ (it has center C). Define $f : G \times G \to N = C \times F$ by $f(x, y) = (1, 1)$ if $x = 1$ or $y = 1$, and $f(x, y) = (1, [x, y])$ otherwise. Let $k : G^3 \to C$ be a 3-cocycle in $\hom_{\mathbb{Z}G}(\bar{F}_G, C)$ representing α. Define a function $L : G \to \operatorname{Aut}(N)$ by

$$L(x)(c, [y, z]) = (c + k(x, y, z), f(x, y)f(xy, z)f(x, zy)^{-1}),$$

and show that it satisfies $L(x)L(y) = \tau(f(x, y))L(xy)$. In particular L induces a homomorphism $\phi : G \to \operatorname{Out}(N)$. Show that the obstruction to realizing (G, N, ϕ) is α. (If you are interested in a proof for the case $G = \mathbb{Z}/2$, see "S. Eilenberg and S. Mac Lane, *Cohomology theory in abstract groups. II. Group extensions with a non-Abelian kernel.* Ann. of Math. (2) **48**, 1947, 326–341").

Exercise 3.4. We saw in the lecture that $H^2(G, C)$ is a functor in two variables. How is this functoriality transposed in the (non-canonical) interpretation of $H^2(G, C)$ as extentions ?

Exercise 3.5. Classify all extentions $1 \to \mathbb{Z}/4 \to E \to \mathbb{Z}/4 \to 1$ up to isomorphism, and give an explicit representative for each isomorphism class.

Exercise 3.6. Classify all extentions $1 \to Q_8 \to E \to \mathbb{Z}/2 \to 1$ up to isomorphisms, and give an explicit representative for each isomorphism class.

Hint: Recall that Q_8 admits the presentation $\langle a, b | a^4 = 1, b^2 = a^2, bab^{-1} = a^3 \rangle$, and show that $\operatorname{Aut}(Q_8) \cong S_4$ and $\operatorname{Out}(Q_8) \cong S_3$, where S_n denotes the group of permutations of a set with n elements.