COHOMOLOGY OF GROUPS : EXERCISES

Sheet 4, 14 Mai 2007

Exercise 4.1. Let $n \geqslant 2$ be an integer, let $G = C_{p^n} = \langle\, t \,|\, t^{p^n} = 1 \,\rangle$ be the cyclic group of order p^n, and let $H < G$ be the subgroup generated by t^p, so that $H \cong C_{p^{n-1}}$. Compute

$$\operatorname{cor}_H^G : H_*(H, M) \to H_*(G, M) \quad \text{and} \quad \operatorname{tr}_H^G : H_*(G, M) \to H_*(H, M)$$

for $M = \mathbb{F}_p$ or $M = \mathbb{Z}$ with the trivial G-action. Compute

$$\operatorname{tr}_H^G : H^*(H, M) \to H^*(G, M) \quad \text{and} \quad \operatorname{res}_H^G : H^*(G, M) \to H^*(H, M)$$

for the same values of G, H and M.

Exercise 4.2. Let G be a finite group, H be a subgroup of G, and let K be any group. Let $\mathbb{F}$ be a field with trivial action of G and K, and consider the Künneth isomorphism

$$\kappa : H^*(G; \mathbb{F}) \otimes_{\mathbb{F}} H^*(K; \mathbb{F}) \xrightarrow{\cong} H^*(G \times K; \mathbb{F}).$$

Show that $\operatorname{res}_{H\times K}^{G\times K} \kappa = \kappa(\operatorname{res}_H^G \otimes \operatorname{id})$, and $\operatorname{tr}_{H\times K}^{G\times K} \kappa = \kappa(\operatorname{tr}_H^G \otimes \operatorname{id})$. Repeat the exercise in homology, for $\operatorname{cor}_{H\times K}^{G\times K}$ and $\operatorname{tr}_{H\times K}^{G\times K}$ (in homology, the hypothesis that G be finite can be supressed, but of course we need $[G : H] < \infty$ for the transfer to be defined).

Exercise 4.3. Repeat exercise 4.1 for $G = C_p^n$ and H a subgroup of order p^{n-1}.

Exercise 4.4. Let G be a group, $x \in G$, and let $\tau_x : G \to G$ be defined by $g \mapsto xgx^{-1}$. Show that $B\tau_x$, as a self map of BG, is homotopic to the identity.

Exercise 4.5. Let G be a group, M be a $\mathbb{Z}G$-module, and take $x \in G$. Show that the homomorphism $c : H_*(G; M) \to H_*(G; M)$ induced by $\tau_x : G \to G$ and $M \to M$, $m \mapsto xm$, is equal to the identity.

Exercise 4.6. Let $K \subset L$ be a finite Galois extension of fields with Galois group G. Show that

$$H^n(G; L) \cong H_n(G; L) \cong \begin{cases} K & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Hint: use the normal basis theorem.