EXERCISES FOR TOPOLOGY III, WS05/06

Sheet 11, January 12 2006

Solutions due on Thursday, 19th of January.

Exercise 11.1. Find an example of a connected, commutative and associative Hopf Algebra A over $\mathbb{Z}$ with two generators, such that A is not isomorphic as an algebra to any tensor product $B \otimes_{\mathbb{Z}} C$ where B and C are connected algebras on one generator.

Exercise 11.2. Repeat the previous exercise, replacing the ground ring $\mathbb{Z}$ by any non-perfect field F.

Exercise 11.3. Show that if A is a connected Hopf-algebra of finite type then $P(A^{\#}) = Q(A)^{\#}$ and $Q(A^{\#}) = P(A)^{\#}$.

Exercise 11.4. Compute $P(A) \to Q(A)$ for the following values of A.
(a) A is a polynomial or exterior Hopf-Algebra on one generator in positive degree,
(b) $A = P(x)/(x^{p^k})$ over a field of characteristic $p > 0$, and $|x| > 0$,
(c) $A = \Gamma(x)$, the divided power algebra over the ground ring $\mathbb{Z}$, $|x| > 0$ even, with coproduct as in Exercise 5.2.

Exercise 11.5. Prove the following Lemma from the lecture : let $A \subset B$ be connected associative Hopf algebras of finite type over a field F, $g : B \to A$ be an A-linear retraction of $A \subset B$, and $\pi : B \to C = F \otimes_A B$ be the projection. Then the map

$$\hat{g} : B \xrightarrow{\Delta} B \otimes B \xrightarrow{g \otimes \pi} A \otimes C$$

is an isomorphism of A-modules.

Exercise 11.6. Let G be a group and k be a field. Show that the diagonal map $G \to G \times G$, $g \mapsto (g, g)$ induces a coproduct on the group-ring kG, and that kG is an ungraded Hopf-algebra. Determine its properties (is it (co-)unital, (co-)associative, (co-)commutative ?). Show that if $H \triangleleft G$, then $kH \triangleleft kH$. Compare $k(G/H)$ to $kG//kH$.