EXERCISES FOR TOPOLOGY III, WS05/06

Sheet 12, January 19 2006

Solutions due on Thursday, 26th of January.

Exercise 12.1. Let $n \geqslant 2$ and let M_n be the Moore space with

$$\tilde{H}_k(M_n;\mathbb{Z}) = \begin{cases} \mathbb{Z}/2 & k = n, \\ 0 & k \neq n, \end{cases}$$

constructed as the homotopy cofibre of a map $S^n \to S^n$ of degree 2. Since $M_n = \Sigma M_{n-1}$, the set $[M_n, M_n]$ of homotopy classes of pointed maps is a group. Show that in this group the order of the identity of M_n in strictly larger than 2.

Hint: view the map $2\mathrm{id}_{M_n}$ as the composition $f \wedge \mathrm{id} : S^1 \wedge M_{n-1} \to S^1 \wedge M_{n-1}$, where f has degree 2. The homotopy cofibre of this map is $\mathbb{R}P^2 \wedge M_{n-1}$. If $2\mathrm{id}_{M_n} \simeq 0$ we would have

$$\mathbb{R}P^2 \wedge M_{n-1} \simeq M_n \vee \Sigma M_n.$$

Show that this is not possible by showing that Sq^2 acts non-trivially on the cohomology of $\mathbb{R}P^2 \wedge M_{n-1}$ (use the Cartan formula and Exercise 2.3, where $Sq^1 = \beta$ was computed for M_n).

Remark: in fact $[M_n, M_n] \cong \mathbb{Z}/4$.

Exercise 12.2. Let u be a class in $H^2(X, A; \mathbb{F}_2)$ such that $Sq^1(u) = 0$. Show that for all $k \geqslant 1$ and $n \geqslant 1$ we have

$$Sq^{2n+1}(u^k) = 0 \quad \text{and} \quad Sq^{2n}(u^k) = \binom{k}{n} u^{k+n}.$$

Exercise 12.3. Let $A(2)$ be the sub-algebra of the Steenrod algebra A_2 generated by Sq^0, Sq^1, Sq^2. Show that $A(2)$ has dimension 8 as $\mathbb{F}_2$-vector space, and describe its multiplicative structure.

Exercise 12.4. Express $Sq^{2^i} Sq^{2^i}$ as a linear combination of admissible monomials.