EXERCISES FOR TOPOLOGY III, WS05/06

Sheet 13, January 26 2006

Solutions due on Monday, 6th of February.

In this sheet we will do computations involving the classifying space $BO(n)$ of the orthogonal group $O(n)$. We recall (or admit) the following facts.
(a) $O(1) \cong \mathbb{Z}/2$, $BO(1) \cong \mathbb{R}P^\infty$, and the inclusion $O(1)^n \to O(n)$ of diagonal matrices induces a continuous map $d : \big(BO(1)\big)^n \to BO(n)$.
(b) The map d induces an injection of cohomology rings

$$d^* : H^*(BO(n);\mathbb{F}_2) \hookrightarrow H^*(BO(1)^n;\mathbb{F}_2),$$

with image the subalgebra of $H^*(BO(1)^n;\mathbb{F}_2) = P(u_1,\dots,u_n)$ generated by the symmetric polynomials $\sigma_1,\dots,\sigma_n$. These are defined by the equation

$$\prod_{i=1}^n (1+u_i) = \sum_{i=0}^n \sigma_i, \quad \text{with} \quad |\sigma_i| = i$$

(hence $\sigma_0 = 1$, $\sigma_1 = u_1 + \cdots + u_n$, $\dots$, $\sigma_n = u_1 \dots u_n$).

Exercise 13.1. Prove that the subalgebra of $P(u_1,\dots,u_n)$ generated by the symmetric polynomials $\sigma_1,\dots,\sigma_n$ is free, so that

$$H^*(BO(n);\mathbb{F}_2) = P(w_1,w_2,w_3,\dots,w_n)$$

with $|w_i| = i$ and $d^*(w_i) = \sigma_i$ (the class w_i is called the i-th Stiefel-Whitney class).

Exercise 13.2. Prove the Wu-formula for the Stiefel-Whitney classes :

$$Sq^j w_i = \sum_{k=0}^j \binom{i-k-1}{j-k} w_{i+j-k} w_k \quad (0 \leqslant j \leqslant i).$$

Here by convention $w_0 = 1$ and $\binom{-1}{0} = 1$.

Exercise 13.3. The space $BO = \bigcup_n BO(n)$ has an H-space structure induced by the direct sum of matrices. Indeed, the map $O(m) \times O(n) \to O(m+n)$, $(A,B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ induces a map $BO(m) \times BO(n) \to BO(m+n)$. Taking the union over m and n we obtain an H-space structure $\mu : BO \times BO \to BO$. Show that $H^*(BO;\mathbb{F}_2) = P(w_1,w_2,w_3,\dots)$, and that the coproduct $\psi = \mu^*$ on $H^*(BO;\mathbb{F}_2)$ is given by

$$\psi(w_k) = \sum_{i=0}^k w_i \otimes w_{k-i}.$$

Exercise 13.4. Admitting that $H_*(BO;\mathbb{F}_2)$ is of finite type, show that there is an isomorphism of algebras

$$H_*(BO;\mathbb{F}_2) \cong P(x_1,x_2,x_3\dots)$$

(with the Pontrjagin product), where x_i is dual to w_1^i with respect to the basis of $H^*(BO;\mathbb{F}_2)$ consisting of monomials in the Stiefel-Whitney classes.