EXERCISES FOR TOPOLOGY III, WS05/06

Sheet 2, 28 October 2005

Solutions due on Monday, 7th of November.

Exercise 2.1. Show that for any pair of spaces (X, A) the diagram

$$\begin{array}{ccc} H^n(A;R) & \xrightarrow{\quad\partial^n\quad} & H^{n+1}(X,A;R) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ \hom_R(H_n(A;R),R) & \xrightarrow{\partial_n^*} & \hom_R(H_{n+1}(X,A;R),R) \end{array}$$

is commutative. Here ∂^n is the connecting homomorphism for cohomology, and ∂_n is the connecting homomorphism for homology.

Exercise 2.2. Let p be a prime number and consider the short exact sequence of $\mathbb{Z}$-modules $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{F}_p \to 0$. Applying $S_*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} -$ to this sequence we obtain a short exact (why ?) sequence of chain complexes

$$0 \to S_*(X;\mathbb{Z}) \xrightarrow{p} S_*(X;\mathbb{Z}) \to S_*(X;\mathbb{F}_p) \to 0.$$

Taking homology we obtain a long exact sequence

$$\cdots \xrightarrow{\partial_{n+1}} H_n(X;\mathbb{Z}) \xrightarrow{i_n} H_n(X;\mathbb{Z}) \xrightarrow{k_n} H_n(X;\mathbb{F}_p) \xrightarrow{\partial_n} H_{n-1}(X;\mathbb{Z}) \xrightarrow{i_{n-1}} \cdots .$$

For all $n \geqslant 1$, we define the mod p Bockstein homomorphism

$$\beta_n : H_n(X;\mathbb{F}_p) \to H_{n-1}(X;\mathbb{F}_p)$$

in homology by $\beta_n = k_{n-1} \circ \partial_n$. Show that
(a) i_n is multiplication by p on the $\mathbb{Z}$-module $H_n(X;\mathbb{Z})$,
(b) β_n is a homomorphism, and it is natural with respect to continuous maps,
(c) $\beta_{n-1} \circ \beta_n = 0$.
Compute the mod p Bockstein homomorphisms for the Moore space $M(n, \mathbb{Z}/p)$, where $n \geqslant 2$.

Exercise 2.3. Define the mod p Bockstein homomorphism

$$\beta^n : H^n(X;\mathbb{F}_p) \to H^{n+1}(X;\mathbb{F}_p)$$

in cohomology (this time apply the functor $\hom_{\mathbb{Z}}(S_*(X;\mathbb{Z}), -)$ to the short exact sequence $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{F}_p \to 0$). Phrase and prove its properties (dual to the properties of the homology Bockstein above), and compute it for the Moore space $M(n, \mathbb{Z}/p)$, $n \geqslant 2$.