EXERCISES FOR TOPOLOGY III, WS05/06

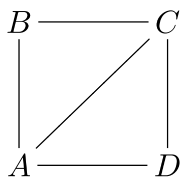
Sheet 4, November 10th 2005

Solutions due on Thursday, 17th of November.

Exercise 4.1. Let π_n be the permutation of the set $\{0, 1, \ldots, n\}$ given by $\pi_n(i) = n - 1$, and let $\varepsilon_n \in \{\pm 1\}$ be its signature. Let $\pi : \Delta^n \to \Delta^n$ be the affine map that sends the vertex e_i to $e_{\pi_n(i)}$. Consider the homomorphism $\theta_n : S_n(X; R) \to S_n(X; R)$ defined on a simplex $\sigma : \Delta^n \to X$ by $\sigma \mapsto \varepsilon_n \sigma \circ \pi_n$, and extended linearly.
(a) Show that $\{\theta_n\}_n$ is a morphism of chain complexes.
(b) Construct an explicit chain homotopy $h_* : S_*(X; R) \to S_{*+1}(X; R)$ between θ_* and the identity.
(c) Use the Theorem of Acyclic Models to show that θ_* is chain homotopic to the identity.

Exercise 4.2. Determine, by explicit computation with the definition of the cup-product, the cohomology algebra $H^*(S^1 \times S^1; \mathbb{Z})$ of the torus.

Here are some suggestions. First find explicit cycles that generate the homology of the torus. Consider the cycles $\sigma = AB$ and $\tau = AD$ in $S_1(S^1 \times S^1; \mathbb{Z})$ and the cycle $z = ABC - ADC$ in $S_2(S^1 \times S^1; \mathbb{Z})$, as pictured below.



(The torus is the quotient of this square obtained by identifying AB with DC and AD with BC.) Show that

$$H_n(S^1 \times S^1; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}\{\bar{\sigma}\} \oplus \mathbb{Z}\{\bar{\tau}\} & \text{if } n = 1, \\ \mathbb{Z}\{\bar{z}\} & \text{if } n = 2, \\ 0 & \text{if } n \geqslant 3. \end{cases}$$

Dualize, and compute the cup product $\bar{\sigma}^* \cup \bar{\tau}^*$, where $\{\bar{\sigma}^*, \bar{\tau}^*\}$ is the basis of $H^1(S^1 \times S^1; \mathbb{Z})$ dual to $\{\bar{\sigma}, \bar{\tau}\}$.

Exercise 4.3. Let $m \geqslant 2$ be an integer. Compute the cohomology algebra

$$H^*(M(\mathbb{Z}/m, 1); \mathbb{Z}/m)$$

of the Moore space $M(\mathbb{Z}/m, 1)$ by a method similar to the one proposed in the previous exercise.