EXERCISES FOR TOPOLOGY III, WS05/06

Sheet 5, November 17th 2005

Solutions due on Thursday, 24th of November.

Exercise 5.1. Let R be a commutative and unital ring, and M, N be R-modules. Describe how the algebras $T(M \oplus N)$, $P(M \oplus N)$ and $E(M \oplus N)$ are related to $T(M)$ and $T(N)$, $P(M)$ and $P(N)$, and $E(M)$ and $E(N)$, respectively.

Exercise 5.2. Consider the polynomial algebra $P(x)$ over $\mathbb{Z}$. Define on $P(x)$ a coproduct $\psi : P(x) \to P(x) \otimes P(x)$ by setting $\psi(x) = 1 \otimes x + x \otimes 1$, and extending it to be a map of $\mathbb{Z}$-algebras. Describe the algebra $\Gamma(x)$, which as $\mathbb{Z}$-module is the dual $\big(P(x)\big)^{\#}$ of $P(x)$, with product induced by the coproduct of $P(x)$. Also describe the coproduct of $\Gamma(x)$ induced by the product of $P(x)$.

Exercise 5.3. Consider a product of spheres $X = S^{n_1} \times \cdots \times S^{n_k}$ with $n_i \geqslant 1$. Describe the algebra $H^*(X;\mathbb{Z})$.

Exercise 5.4. Let (X, x_0) and (Z, y_0) be based spaces, and let $X \vee Y$ be their wedge, i.e. $X \vee Y = (X \sqcup Y)/(x_0 = y_0)$. Show that there is an isomorphism of (non-unital) R-algebras

$$\tilde{H}^*(X;R) \oplus \tilde{H}^*(Y;R) \cong \tilde{H}^*(X \vee Y;R),$$

where the product on the left is component-wise.

Exercise 5.5. Show that if $m, n \geqslant 1$, then the spaces $S^m \times S^n$ and $S^m \vee S^n \vee S^{m+n}$ are not homotopy equivalent.

Exercise 5.6. Let M_g be an orientable surface of genus g. Compute the cohomology algebra $H^*(M_g;\mathbb{Z})$.

Hint: M_g is the connected sum of S^2 with g tori $T_i = S^1 \times S^1$, $i = 1, \dots, g$, i. e.

$$M_g = \big[\big(S^2 \smallsetminus (D_1^\circ \sqcup \cdots \sqcup D_g^\circ)\big) \sqcup (T_1 \smallsetminus \bar{D}_1^\circ) \sqcup \cdots \sqcup (T_g \smallsetminus \bar{D}_g^\circ)\big]/\sim$$

Here $D_1, \dots, D_g$ are g two-dimensional disks disjointly embedded in S^2, $\bar{D}_i$ is a disk embedded in T_i, and the superscript $(-)^\circ$ denotes the interior. The relation $\sim$ identifies ∂D_i with $\partial \bar{D}_i$ for $1 \leqslant i \leqslant g$. Assume $g \geqslant 1$. The space $X = S^2 \smallsetminus (D_1^\circ \sqcup \cdots \sqcup D_g^\circ) \subset M_g$ is homotopy equivalent to $\bigvee_{g-1} S^1$. The quotient of M_g by X is homeomorphic to $\bigvee_g T$. Use the cofibration up to homotopy

$$\bigvee_{g-1} S^1 \to M_g \to \bigvee_g T$$

and the induced long exact sequence in cohomology to solve the exercise.

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