EXERCISES FOR TOPOLOGY III, WS05/06

Sheet 6, November 24th 2005

Solutions due on Thursday, 1st of November.

Exercise 6.1. Compute the cohomology algebra $H^*(\mathbb{C}P^n;\mathbb{Z})$ for $n \geqslant 1$.

Exercise 6.2. A *division R-algebra structure* on an R-module A is an R-bilinear homomorphism $\mu : A \times A \to A$ (not necessarily unital, associative, nor commutative), such that for any $a \in A\backslash\{0\}$ the homomorphisms $\mu(a,-)$ and $\mu(-,a) : A \to A$ are surjective.

Show that if $\mathbb{R}^n$ is a division $\mathbb{R}$-algebra, then n is a power of 2.
Hint: assume that $n \geqslant 2$ and that we have a division $\mathbb{R}$-algebra structure on $\mathbb{R}^n$ given by $\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. We then have a map $S^{n-1} \times S^{n-1} \to S^{n-1}$ given by $(x,y) \mapsto \mu(x,y)/|\mu(x,y)|$. This induces a map $f : \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$. Show that the homomorphism of $\mathbb{F}_2$-algebras

$$(\times)^{-1} \circ f^* : H^*(\mathbb{R}P^{n-1};\mathbb{F}_2) \to H^*(\mathbb{R}P^{n-1};\mathbb{F}_2) \otimes H^*(\mathbb{R}P^{n-1};\mathbb{F}_2)$$

satisfies $\alpha \mapsto \alpha \otimes 1 + 1 \otimes \alpha$, where α is the generator of $H^1(\mathbb{R}P^{n-1};\mathbb{F}_2)$. Deduce from $\alpha^n = 0$ that n is a power of 2.

Exercise 6.3. Show that if a space X is the union of n open contractible subspaces, then all n-fold cup products $\alpha_1 \cup \cdots \cup \alpha_n$ with $|\alpha_i| \geqslant 1$ are 0 in $H^*(X;R)$.
Hint: Induction on n, using the relative cup product.

Exercise 6.4. Using the cup product, show that if $m,n \geqslant 1$ then any map $S^{m+n} \to S^m \times S^n$ induces the zero homomorphism $H_{m+n}(S^{m+n};\mathbb{Z}) \to H_{m+n}(S^m \times S^n;\mathbb{Z})$.

Exercise 6.5. Find a connected space X so that for any field F, the cross product

$$H^*(X;F) \otimes H^*(X;F) \xrightarrow{\times} H^*(X \times X;F)$$

is not an isomorphism