EXERCISES FOR TOPOLOGY III, WS05/06

Sheet 7, December 1st 2005

Solutions due on Thursday, 8th of December.

Exercise 7.1.
(a) Let $\cap : S_*(X) \otimes_R S^*(X) \to S_*(X)$ be defined as $\cap = (1 \otimes \langle -, - \rangle) \circ \big((\mathrm{AW}\Delta_*) \otimes 1\big)$, and assume that the cup product on the cochains level is also defined using AW. Prove that if $\sigma \in S_{p+q}(X)$, $\alpha \in S^p(X)$ and $\beta \in S^q(X)$, then the formula

$$\langle \beta, \sigma \cap \alpha \rangle = \langle \beta \cup \alpha, \sigma \rangle$$

holds. Deduce that $\cap$ makes $S_*(X)$ into a left $S^*(X)$-module and $H_*(X)$ into a left $H^*(X)$-module. State and prove the corresponding results for relative homology and cohomology.
(b) Show that for any $z \in H_{p+q}(X, A)$ the diagram

$$\begin{array}{ccc} H^p(X,A) & \longrightarrow & H^p(X) \\ {\scriptstyle z\cap -}\downarrow & & \downarrow{\scriptstyle z\cap -} \\ H_q(X) & \longrightarrow & H_q(X,A) \end{array}$$

is commutative

Exercise 7.2. Let $X = S^{n_1} \times \cdots \times S^{n_k}$ for $k \geqslant 1$ and $n_j \geqslant 1$. Let M_g be the orientable surface of genus g. Compute
(a) $H_*(X; \mathbb{Z})$ as a module over $H^*(X; \mathbb{Z})$,
(b) $H_*(M_g; \mathbb{Z})$ as a module over $H^*(M_g; \mathbb{Z})$,
(c) $H_*(\mathbb{R}P^n; \mathbb{F}_2)$ as a module over $H^*(\mathbb{R}P^n; \mathbb{F}_2)$, and
(d) $H_*(\mathbb{C}P^n; \mathbb{Z})$ as a module over $H^*(\mathbb{C}P^n; \mathbb{Z})$.

For all these cases, deduce that if m is maximal such that H_m is non-zero, then H_m is free of rank one on a generator z and $z \cap - : H^k \to H_{m-k}$ is an isomorphism for any k. Show that the space in question is a compact topological manifold of dimension m.

Exercise 7.3. Let $n \geqslant 1$. Show that $\mathbb{R}P^n$ is $\mathbb{Z}$-orientable if and only if n is odd.

Exercise 7.4. Show that R-orientations of a topological manifold M are in 1-1 correspondence with sections $s \in \Gamma(M)$ such that $s(M) \subset \varepsilon^{-1}(1)$.