EXERCISES FOR TOPOLOGY III, WS05/06

Sheet 8, December 8th 2005

Solutions due on Thursday, 15th of December.

In this exercise sheet, we work out some of the details in the proof of the Poincaré Duality Theorem.

Exercise 8.1. Show that if M is R-oriented of dimension n, and if $U \subset V \subset M$ are open subsets, then the diagram

$$\begin{tikzcd} H^q_c(U) \arrow[r, "D"] \arrow[d, "i_*"'] & H_{n-q}(U) \arrow[d, "i_*"] \\ H^q_c(V) \arrow[r, "D"] & H_{n-q}(V) \end{tikzcd}$$

is commutative, with i_* induced by the inclusion $U \subset V$.

Exercise 8.2. Prove the Poincaré Duality Theorem for $\mathbb{R}^n$.

Exercise 8.3. Consider an open triad (X, U, V), i.e. X is a space and U and V are open subsets. Construct and prove the exactness of the relative Mayer-Vietoris sequence

$$\xrightarrow{\partial} H_q(X, U \cap V) \to H_q(X, U) \oplus H_q(X, V) \to H_q(X, U \cup V) \xrightarrow{\partial} H_{q-1}(X, U \cap V) \to$$

Similarly, prove the existence of the relative Mayer-Vietoris sequence in cohomology

$$\xleftarrow{\delta} H^q(X, U \cap V) \leftarrow H^q(X, U) \oplus H^q(X, V) \leftarrow H^q(X, U \cup V) \xleftarrow{\delta} H^{q-1}(X, U \cap V) \leftarrow$$

Exercise 8.4. Let U and V be open subsets of X, and consider the relative Mayer-Vietoris sequences of the triad (X, U, V), as constructed in the previous exercise. Show that if $a \in H_{n+1}(X, U \cup V)$ and if $\beta \in H^q(X, U \cap V)$ then the equality

$$\partial(a) \cap \beta = \pm a \cap \delta(\beta)$$

holds in $H_{n-q}(X)$.

Exercise 8.5. Prove that taking the colimit over a directed set of indices is an exact functor. Find an example where the colimit functor (over a non-directed indexing category) is not exact.