EXERCISES FOR TOPOLOGY III, WS05/06

Sheet 9, December 15th 2005

Solutions due on Thursday, 22th of December.

Exercise 9.1. Show that the $\mathbb{Z}$-orientable manifold $\mathbb{C}P^{2n}$ does not admit any self homotopy equivalence $f : \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$ which reverses a given $\mathbb{Z}$-orientation.

Exercise 9.2. Let M be a compact manifold of dimension n, such that M is homotopy equivalent to a suspension ΣY for some path-connected space Y. Show that $H^*(M;\mathbb{Z}) \cong H^*(S^n;\mathbb{Z})$. Can you find a continuous map that realizes this isomorphism ?

Exercise 9.3. Prove the following result on colimits (which we used in the proof of the Poincaré Duality Theorem). Let I and J be directed sets, and assume that J has a filtration $\{J_i\}_{i\in I}$ with $J_i \subseteq J_{i'}$ whenever $i \leqslant i'$ and $J = \bigcup_{i\in I} J_i$. Given a functor $F : J \to R-$modules, show that the natural map

$$\operatorname*{colim}_{i\in I} \operatorname*{colim}_{j\in J_i} F(j) \to \operatorname*{colim}_{j\in J} F(j)$$

is an isomorphism.

Exercise 9.4. Let A be the "Hawaiian earrings", viewed as the compact subset of $S^2 = \mathbb{R}^2 \cup \{\infty\}$ defined by $A = \bigcup_{n\geqslant 1} A_n$, where A_n is the circle in $\mathbb{R}^2$ with center $(1/n, 0)$ and radius $1/n$.
(a) Compute the first Čech-cohomology group of A from the definition, and verify that your result is compatible Alexander Duality.
(b) Show that the natural map $\check{H}^1(A;\mathbb{Z}) \to H^1(A;\mathbb{Z})$ is not surjective.

Exercise 9.5. Let $Y \subset \mathbb{R}^2$ be the subspace $Y = \{\big(x, \sin(1/x)\big) \in \mathbb{R}^2 | x \neq 0\}$, and let A be its closure in $S^2 = \mathbb{R}^2 \cup \{\infty\}$.
(a) Some question as (a) in the previous exercise.
(b) Show that the natural map $\check{H}^1(A;\mathbb{Z}) \to H^1(A;\mathbb{Z})$ is not injective.