EXAM

30 October 2018

Exercise 1. State in more precise terms and prove the following slogan : *In any category $\mathcal{C}$, a retract of a push-out square is a push-out square.*

Exercise 2. Let R be a commutative unital ring, and let Mod_R be the category of R-modules.
(a) Recall a construction of the push-out in the category Mod_R.
(b) Does an analogue of the Van-Kampen Theorem (group version) holds for the functor $H_1(-;R) : \mathrm{Top} \to \mathrm{Mod}_R$? In other words, does this functor preserve push-outs (under the same condition) ?
(c) What about $H_n(-;R)$ for $n \geqslant 2$?

Exercice 3. Let $(\mathrm{Mod}_R)^{\mathbb{N}}$ be the category of functors $F : \mathbb{N} \to \mathrm{Mod}_R$; we will picture such a functor as a diagram $F(0) \xrightarrow{f_0} F(1) \xrightarrow{f_1} F(2) \xrightarrow{f_2} \dots$. Note that $(\mathrm{Mod}_R)^{\mathbb{N}}$ is obviously an additive category, with zero-object the constant functor with value 0. A sequence $0 \to F \to G \to H \to 0$ in $(\mathrm{Mod}_R)^{\mathbb{N}}$ is called *exact* if for all $n \in \mathbb{N}$, the sequence evaluated at n is exact in Mod_R.
(a) Let $F \in (\mathrm{Mod}_R)^{\mathbb{N}}$. Give a justification why $\operatorname{colim} F := \operatorname{colim}_{n\in\mathbb{N}} F$, the colimit of F, is naturally isomorphic to

$$\operatorname{coker}\left(\bigoplus_{m\in\mathbb{N}} F(m) \xrightarrow{\psi_F} \bigoplus_{n\in\mathbb{N}} F(n)\right),$$

where ψ_F is given on the summand $F(m)$ by $x \mapsto x - f_m(x)$.
(b) Prove that for any exact sequence $0 \to F \to G \to H \to 0$ in $(\mathrm{Mod}_R)^{\mathbb{N}}$ we have an exact sequence

$$0 \to \operatorname{colim} F \to \operatorname{colim} G \to \operatorname{colim} H \to 0\,.$$

You may use the Snake Lemma without proving in.

Exercice 4. For $k \in \mathbb{N}$, let $t_k : S^1 \to S^1$ be a chosen map of degree k.
(a) Let $X : \mathbb{N} \to \mathrm{Top}$ be a functor such that for all n, $X(n) \to X(n+1)$ is an inclusion of a subspace. Let $X(\infty) = \operatorname{colim} X$. We identify $X(n)$ with its image in $X(\infty)$. We assume that for any compact subspace $K \subset X(\infty)$, there exists $n \in \mathbb{N}$ with $K \subset X(n)$ (this is in fact always true). Prove that for all $m \in \mathbb{N}$, $H_m(X(\infty);R) = \operatorname{colim}_n H_m(X(n);R)$.

(b) Let $Y : \mathbb{N} \to \mathrm{Top}$ by defined by

$$Y(n) = \left(\left(\coprod_{k=0}^{n-1} S^1 \times I_k \right) \sqcup S^1 \times [n, n+\tfrac{1}{2}) \right) / \sim \ .$$

Here $I_k = [k, k+1] \subset \mathbb{R}$, and $\sim$ is the equivalence relation generated by

$$S^1 \times I_k \ni (x, k+1) \sim (t_{k+1}(x), k+1) \in S^1 \times I_{k+1} \quad \text{for } k \leqslant n-2, \text{ and}$$
$$S^1 \times I_{n-1} \ni (x, n) \sim (t_n(x), n) \in S^1 \times [n, n+\tfrac{1}{2})\,.$$

The morphisms $y_n : Y(n) \to Y(n+1)$ are the obvious inclusions. Show that the map $i_n : S^1 \to Y(n)$, $x \mapsto (x, n)$ is a homotopy equivalence, and that $y_n \circ i_n \simeq i_{n+1} \circ t_{n+1}$. Then, with the help of (a), compute $H_*(Y(\infty); \mathbb{Z})$.

(c) Deduce from the following example that homology does not commute with arbitrary sequential colimits : for all $n \in \mathbb{N}^*$, let $A_n \subset S^1$ be the closed arc that is the image of the closed interval $[\frac{1}{n+1}, 1 - \frac{1}{n+1}] \subset [0,1]$ under $[0,1] \to S^1$, $x \mapsto e^{2\pi i x}$. Consider $Z : \mathbb{N}^* \to \mathrm{Top}$, $Z(n) = S^1/A_n$, with $Z(n) \to Z(n+1)$ the obvious quotient map.

Exercise 5. Suppose that X is a space with finitely generated homology, meaning that the direct sum of its homology groups $H_*(X;\mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module (in particular $H_n(X;\mathbb{Z}) = 0$ for n large enough). We then define the *Euler characteristic of* X as the integer

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n \mathrm{rank}\big(H_n(X;\mathbb{Z})\big)\,,$$

where for a finitely generated $\mathbb{Z}$-module A, expressed as sum of cyclic groups, $\mathrm{rank}(A)$ is the number of $\mathbb{Z}$-summands, or, equivalently, $\mathrm{rank}(A) = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})$.

(a) Compute $\chi(S^n)$, $\chi(\mathbb{R}P^n)$ and $\chi(\mathbb{C}P^n)$ for all $n \geqslant 1$.

(b) Show that if X is a space with finitely generated homology, there exists a chain complex E_* of free $\mathbb{Z}$-modules, with the direct sum $\bigoplus_n E_n$ finitely generated, and chain-homotopy equivalent to $S_*(X;\mathbb{Z})$.

Hint : chose a finitely generated sub-module $F_n \subset Z_n(X)$ such that

$$i_n : F_n \subset Z_n(X) \to H_n(X;\mathbb{Z})$$

is surjective. Let $K_n = \ker(i_n)$, and $E_n = F_n \oplus K_{n-1}$, with a suitable boundary operator. There exists $h_n : K_n \to S_{n+1}(X;\mathbb{Z})$ with $dh_n(a) = a$ for all $a \in K_n$, which can then be used to define a chain homotopy equivalence $E_* \to S_*(X;\mathbb{Z})$.

(c) Show that, in the notations of (b), $\chi(X) = \sum_{n=0}^{\infty} (-1)^n \mathrm{rank}(E_n)$.

(d) Deduce that for any field F, we have $\chi(X) = \sum_{n=0}^{\infty} (-1)^n \dim_F H_n(X;F)$.

(e) If X and Y are spaces with finitely generated homology, prove that

$$\chi(X \sqcup Y) = \chi(X) + \chi(Y) \ \text{ and } \ \chi(X \times Y) = \chi(X)\chi(Y)\,.$$

(f) Using Poincaré duality and (d), prove that if M is a closed manifold of odd dimension, then $\chi(M) = 0$.

(g) If S and T are two surfaces, let $S\#T$ be their connected sum (remove a small open disk in a coordinate neighborhood in S, and in T, and glue along the boundary). Compute $\chi(S\#T)$.

(h) Deduce $\chi(\mathbb{R}P^2 \# \mathbb{R}P^2)$.

(i) Using (g) or a computation in homology, compute $\chi(M_g)$, for M_g the closed orientable surface of genus g.