GEOMETRY AND TOPOLOGY

Sheet 1, 27.09.2023

Exercise 1.1. Let X and Y be topological spaces, and let $A \subset X$ and $B \subset Y$ be subspaces. Show that $A \times B \subset X \times Y$ is a subspace.

Exercise 1.2. Let X be a totally ordered set; the order topology is defined by a sub-basis consisting of $L_a = \{x \in X; x > a\}$ and $S_a = \{x \in X; x < a\}$ for $a \in X$, just as for the real numbers. A subset $A \subset X$ inherits a total order from X. Find different examples showing that

- the subspace topology of A (relative the order topology of X), and
- the order topology of A,

sometimes agree, sometimes not. Find a condition of the subest A of the orderet set X so that these two notions agree.

Exercise 1.3. Consider the usual order on $\mathbb{R}$, and endow $\mathbb{R} \times \mathbb{R}$ with the lexicographical order. Let X be the space defined as the set $\mathbb{R} \times \mathbb{R}$ with the corresponding lexicographical order topology. Show that X is homeomorphic to the product space $\mathbb{R} \times \mathbb{R}^{\delta}$, where $\mathbb{R}$ has the usual topology, and $\mathbb{R}^{\delta}$ is $\mathbb{R}$ with the discrete topology.

Exercise 1.4. Complete the proof of Proposition 1.11 from the lecture (existence and uniqueness of the quotient topology).

Exercise 1.5. Suppose $p : X \to Y$ and $i : Y \to X$ are continuous maps such that $p \circ i = \mathrm{id}_Y$. Show that p is a quotient map.
Remark: If $i : Y \to X$ is the inclusion of a subspace, we say that p is *a retraction of X onto Y*.

Exercise 1.6. We define the (2-dimensional) torus as the product $T^2 = S^1 \times S^1$.

(a) Construct a continuous, surjective map $[0,1]^2 \to T^2$, and use it to show that T^2 is homeomorphic to a quotient Q of $[0,1]^2$.
(b) Define an embedded torus M in $\mathbb{R}^3$ (for example, take the subspace of $\mathbb{R}^3$ corresponding to the set of points on the trajectory of the circle of radius 1 in the plane Oxy, of center $(2,0,0)$, under the rotations of axis $0z$). Then, define a continus surjective map $[0,1]^2 \to M$, and deduce that M is homeomorphic to Q and T^2.

Exercise 1.7. We define the closed 2-dimensional disk D^2 as the subspace $D^2 = \{x \in \mathbb{R}^2 ; \|x\| \leq 1\}$. Produce a surjective map $D^2 \to S^2$ that induces a homeomorphism $D^2/S^1 \to S^2$.

Exercise 1.8. Prove that $S^{n-1} \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^n \setminus \{0\}$.

Exercise 1.9. A map $f : X \to Y$ is an *open map* if for every open subset U of X the set $f(U)$ is open in Y. Show that the projection maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are open maps.

Exercise 1.10. Let X and Y be topological spaces. Let $X \coprod Y$ denote the disjoint union of the sets X and Y equipped with the canonical inclusion maps $i_X : X \to X \coprod Y$ adn $i_Y : Y \to X \coprod Y$. Prove that there exists a unique topology $\tau_{X \coprod Y}$ on the set $X \coprod Y$ such that the map

$$i : \mathrm{Top}(X \coprod Y, Z) \to \mathrm{Top}(X, Y) \times \mathrm{Top}(Y, Z)$$

given by $i(f) = (f \circ i_X, f \circ i_Y)$ is a bijection of sets.

Exercise 1.11. Given topolgical spaces X and Y and points $x \in X$ and $y \in Y$, define $X \vee Y$ as the quotient of $X \coprod Y$ by the equivalence relation generated by $x \sim y$. Formulate and prove a universal property for the topology on $X \vee Y$.