M1 de Mathématiques — Institut Galilée, 2023-2024

GEOMETRY AND TOPOLOGY

Sheet 2, 04.10.2023

Exercise 2.1. Write out the proofs of Proposition 1.22, Lemma 1.23, and Proposition 1.30.

Exercise 2.2. Let I be a set and let X be a topological space. For each $i \in I$, let $X_i := X$. Prove that there is a homeomorphism $\coprod_{i \in I} X_i \to I^\delta \times X$, where I^δ denotes the set I with the discrete topology.

Exercise 2.3. We define the Cantor set as $C = \cap_{n=0}^{\infty} C_n$ where $C_0 = [0,1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and more generally $C_n := \frac{1}{3} \cdot C_{n-1} \cup (\frac{2}{3} + \frac{1}{3} \cdot C_{n-1})$ for $n \geq 1$. We equip $C \subset \mathbb{R}$ with the subspace topology where $\mathbb{R}$ has the usual topology. Prove that the product $\prod_{\mathbb{N}} \{0,1\}$ is homeomorphic to the Cantor set.

Exercise 2.4. Suppose X is a topological space.

(a) Show that connected components are closed subspaces.
(b) Deduce that if X has finitely many connected components, then X is homeomorphic to the coproduct of its connected components.

Exercise 2.5. Describe the connected components of $\mathbb{Q}$ regarded as a subspace of $\mathbb{R}$ where $\mathbb{R}$ has the usual topology.

Exercise 2.6. Consider the commutative diagram of topological spaces and continuous maps

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y'. \end{array}$$

(1) Suppose the diagram is a pushout and $X \to Y$ is a homeomorphism. What can you say about $X' \to Y'$? Is the diagram also a pullback in this case ?
(2) Formulate and prove an analoguous result when the diagram is a pullback.

Exercise 2.7. Consider the commutative diagram of topological spaces and continuous maps

$$\begin{array}{ccccc} X & \longrightarrow & X' & \longrightarrow & X'' \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Y' & \longrightarrow & Y'' \end{array}$$

and prove the following:

(1) Suppose the right-hand-square is a pullback. Prove that the left-hand-square is a pullback if and only if the outer square is a pullback.
(2) Suppose the left-hand square is a pushout. Prove that the right-hand square is a pushout if and only if the outer square is a pushout.

Exercise 2.8. Let $S^n = \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$. Let

$$D^n_N = \{x \in S^n \,;\, x_{n+1} \geq 0\}, \quad D^n_S = \{x \in S^n \,;\, x_{n+1} \leq 0\} \quad \text{and} \quad D^n = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$$

be the northern, resp. southern hemisphere of S^n, and the n-dimensional disc.

(a) Explain why S^n is the pushout of the inclusions $D_s^n \leftarrow D_S^n \cap D_N^n \rightarrow D_N^n$.
(b) Construct homeomorphisms h_N, h_S and h_{eq} making the following diagram commute (where horizontal maps are inclusions),

$$\begin{array}{ccccc} D_N^n & \longleftarrow & D_N^n \cap D_S^n & \longrightarrow & D_S^n \\ \downarrow h_N & & \downarrow h_{\mathrm{eq}} & & \downarrow h_S \\ D^n & \longleftarrow & S^{n-1} & \longrightarrow & D^n \end{array}$$

and deduce that the pushout of the lover line is homeomorphic to S^n. (See Example 1.34.)

Exercise 2.9. Suppose X, Y, and Z are connected nonempty topological spaces. Consider a pushout diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

of topological spaces and continuous maps. Is W connected?