GEOMETRY AND TOPOLOGY

Sheet 3, 11.10.2023

Exercise 3.1. Let $p : X \to Y$ be a quotient map of spaces, with Y connected. Assume furthermore that for any $y \in Y$, the subspace $p^{-1}(\{y\}) \subset X$ is connected. Show that X is then connected.

Exercise 3.2. Give an example of a quotient map that is neither open nor closed.

Exercise 3.3. Let $X = \prod_{i=1}^n X_i$ be a product of spaces, and $p_i : X \to X_i$ the projection for $i \geq 1$.

(a) Show that p_i is an open map, but not necessarily a closed map.
(b) Show that if X_i is non-empty for $i > 1$, then p_1 is a quotient map.

Exercise 3.4. Let $\mathbb{R}$ and $\mathbb{R}^2$ be given the Euclidean topology. Consider the following spaces A, B, C and D, that can each be seen as "an infinite union of circles":

(a) A is the union of circles with center $(0, n)$ and radius n in $\mathbb{R}^2$, for all $n \geq 1$, with the subspace topology;
(b) B is the union of circles with center $(0, \frac{1}{n})$ and radius $\frac{1}{n}$ in $\mathbb{R}^2$ for all $n \geq 1$, with the subspace topology;
(c) $C = \mathbb{R}/\mathbb{Z}$, the quotient space of $\mathbb{R}$ by the *subspace* $\mathbb{Z}$ (not the quotient of groups!);
(d) $D = \bigvee_{\mathbb{N}} S^1$, where $\vee$ denotes the coproduct of pointed spaces (see Exercise 1.11).

For which pairs (X, Y), where X and Y run through the above spaces, does there exist a continuous bijection $X \to Y$? Which pairs (X, Y) consist of homeomorphic spaces ?

Exercise 3.5. Show that the product of the quotient maps $\mathrm{id} : \mathbb{Q} \to \mathbb{Q}$ and $q : \mathbb{R} \to \mathbb{R}/\mathbb{N}_{\geq 1}$ (quotient by the *subspace*) is not a quotient map. (Hint: For each $n \geq 1$, let $c_n = \sqrt{2}/n$ and consider the straight lines in $\mathbb{R}^2$ with slope 1 and -1 respecitively through the point (n, c_n). Then let U_n consist of the points of $\mathbb{R} \times \mathbb{Q}$ that sit above both of these lines or below both of these lines, and also between the vertical lines $x = n - \frac{1}{4}$ and $x = n + \frac{1}{4}$. Finally, consider $U' = p \times \mathrm{id}(\cup_{\mathbb{N}_{\geq 1}} U_n)$.)

Exercise 3.6. Let G be a topological group, and H be a subgroup. Show that the quotient G/H of groups (i.e. the space of orbits) is Hausdorff if and only if H is closed. What does it say in the case $H = \{e\}$?

Exercise 3.7. Show that for $n \in \mathbb{N}$ and $n \geq 1$, the quotient of $\mathbb{R}^n$ by its subgroup $\mathbb{Z}^n$ is homeomorphic to the n-torus $T_n := (S^1)^n$.