GEOMETRY AND TOPOLOGY

Sheet 4, 18.10.2023

Exercise 4.1. This is a variation on Exercise 2.8, and an illustration that in Proposition 1.35, we don't just need homeomorphisms, but compatible ones (the diagram in the hypothesis of Proposition 1.35 must commute).

(a) Show that the push-out of the diagram $\mathbb{R}^n \xleftarrow{i} \mathbb{R}^n \setminus \{0\} \xrightarrow{h} \mathbb{R}^n$, where i is the inclusion and $h(x) = \frac{x}{\|x\|^2}$, is homeomorphic to S^n.

(b) Show that the push-out P of the diagram $\mathbb{R}^n \xleftarrow{i} \mathbb{R}^n \setminus \{0\} \xrightarrow{i} \mathbb{R}^n$ is a non-compact, non-Hausdorff space that is second-countable and locally homeomorphic to $\mathbb{R}^n$. What do you deduce about the definition of an n-manifold ?

Exercise 4.2. Let $U \subset \mathbb{R}^n$ be an open subset and let $f : U \to \mathbb{R}^k$ be a continuous function. Consider the graph of f, given by the set

$$\Gamma(f) = \{(x, y) : x \in U, y = f(x)\} \subset \mathbb{R}^n \times \mathbb{R}^k$$

equipped with the subspace topology. Prove that $\Gamma(f)$ is a m-dimensional manifold and determine the dimension m.

Exercise 4.3. Let X and Y be spaces

(a) If Y is a subspace of X, show that the inclusion $Y \to X$ is an embedding.
(b) If $f : Y \to X$ is a continuous injection that is either open or closed, then f is an embedding.
(c) Give an example of an embedding that is neither an open nor a closed map
(d) Give an example of a continuous injection that is not an embedding.

Exercise 4.4. A *Lie group* is a toplogical manifold M that is endowed with the structure of a topological group. Show that for all $n \geq 1$, $GL_n(\mathbb{R})$ is a Lie group.

Exercise 4.5. Provide details in showing that D^n is an n-manifold with boundary.

Exercise 4.6. Use the *Invariance of Domain* to prove the following results:

(a) If a topological space is non-empty and is both an m-manifold and an n-manifold, then $m = n$.
(b) If M is an n-manifold with boundary, then ∂M is an $(n-1)$-manifold without boundary.
(c) Let M and N be topological n-manifolds, and assume that M is compact without boundary and N is connected. Then any embedding $M \to N$ must be surjective (and hence a homeomorphism).