GEOMETRY AND TOPOLOGY

Sheet 5, 25.10.2023

Homework: solve all the exercises below and turn them in at the lecture on November 7th.

Exercise 5.1. Complete the proofs of Proposition 2.18 and 2.25 by providing in full details proofs of the following facts (either directly, or proving and applying a more general result).

(a) The push-out $M \coprod_h N$ obtained by glueing ∂M and ∂N along a homeomorphism h is second-countable and Hausdorff.

(b) The space $\mathbb{R}P^n$ is second-countable and Hausdorff.

Exercise 5.2. Let $n \geq 2$, and let P be a regular polygon with $2n$ edges in $\mathbb{R}^2$ (including its interior), and let Q be a quotient of P obtained by identifying each edge with exactly one other edge (by preserving the orientation or not). Show that Q is a connected, closed (i.e. compact and without boundary) surface.

Exercise 5.3. In the situation of the previous exercise, assuming that Q is not a sphere, show that by cutting and pasting, you can replace P by P' with identifications that produce surface Q' homeomorphic to Q, such that all vertices of P' are identified to a single point of P'.

Exercise 5.4. Show that gluing a Möbius band and a disk along their boundary is a surface homeomorphic to $\mathbb{R}P^2$. What is the result of gluing two Möbius bands along their boundaries ?

Exercise 5.5. Show that $T^2 \# \mathbb{R}P^2$ is homeomorphic to $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

Exercise 5.6. The goal of this exercise is to how that $\mathbb{R}P^3$ is homeomorphic to $SO(3)$, the group of rotations of the Euclidean space $\mathbb{R}^3$ (with the subspace topology $SO(3) \subset M_3(\mathbb{R}) = \mathbb{R}^9$).

(a) Show (as in the lecture for $n = 2$) that $\mathbb{R}P^n$ is homeomorphic to the quotient $D^n/\sim$ of the n-disk D^n obtained by identifying antipodal points of its boundary S^{n-1}.

(b) Show that the map $f : D^3 \to SO(3)$ defined by mapping $x \neq 0$ to the rotation of axis $\langle x \rangle$ and angle $\|x\| \cdot \pi$ (make this precise by using the orientation), and 0 to $\mathrm{id}_{\mathbb{R}^3}$, is continuous, and induces a continuous, bijective map $\bar{f} : D^3/\sim \to SO(3)$. Conclude.