GEOMETRY AND TOPOLOGY

Sheet 6, 07.11.2023

Exercise 6.1. Let G and H be two groups, considered as categories $\mathbf{G}$ and $\mathbf{H}$ with one object.

(a) What is a functor $F : \mathbf{G} \to \mathbf{H}$?

(b) Given two functors $F_1, F_2 : \mathbf{G} \to \mathbf{H}$, under which conditions does a natural transformation $\eta : F_1 \to F_2$ exist ?

Exercise 6.2. Let **Grp** be the category of groups and group homomorphisms, and let **Ab** be the full subcategory of Abelian groups. For a group G, let $[G, G]$ be the *subgroup of commutators of* G, which is the subgroup generated by the commutators $[g, h] = ghg^{-1}h^{-1}$ for all $g, h \in G$. Prove the following statements:

(a) $[G, G]$ is a normal subgroup of G, and $G^{\mathrm{ab}} = G/[G, G]$ is an Abelian group.

(b) The correspondence $G \mapsto G^{\mathrm{ab}}$ is part of a functor of a functor $(-)^{\mathrm{ab}} : \mathbf{Grp} \to \mathbf{Ab}$ with

$$\mathbf{Ab} \xrightarrow{\subset} \mathbf{Grp} \xrightarrow{(-)^{\mathrm{ab}}} \mathbf{Ab}$$

the identity.

Exercise 6.3. Let G be a group and let

$$Z(G) = \{g \in G \mid hg = gh \text{ for all } h \in G\}$$

be the center of G. It is clear that $Z(G)$ is an Abelian subgroup of G is. Prove that there is no functor $F : \mathbf{Grp} \to \mathbf{Ab}$ with $F(G) = Z(G)$ for all group G.

Hint: Consider a factorization of the identity $\Sigma_2 \to \Sigma_3 \to \Sigma_2$, where Σ_n denotes the symmetric group with $n!$ elements denotes.

Exercise 6.4. Show that in the category **set** of sets, the property of a map of being *injective* (resp. *surjective*) can be defined in terms of morphisms of **set**. In a general category $\mathbf{C}$, we call morphisms having this property *monomorphisms* (resp. *epimorphisms*).

Exercise 6.5. Let **ring** be the category of rings (with unit) and homomorphisms of rings. Let $f : \mathbb{Z} \to \mathbb{Q}$ be be the inclusion.

(a) Is f a monomorphism in **ring**?

(b) Is f an epimorphism in **ring**?

(c) Is f an isomorphism in **ring**?

Exercise 6.6. Let $(P, \leq)$ and $(Q, \leq)$ be two quasi-ordered sets.

(a) Show that there is a small category $\mathbf{P}$ with P as set of objects, and a unique morphism $a \to b$ iff $a \leq b$.

(b) Describe the functors $\mathbf{P} \to \mathbf{Q}$.

(c) Given two functors $F, G : \mathbf{P} \to \mathbf{Q}$, under what conditions does a natural transformation $F \Longrightarrow G$ exist? How many such transformations are there ?

Exercise 6.7. Consider the following subspaces of $\mathbb{R}$:

$$\mathbb{Z},\ \mathbb{Q},\ \ X = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \dots\},\ \ Y = X \cup \{0\}\,.$$

(a) Which are homeomorphic ?

(b) Which have the same homotopy type ?