GEOMETRY AND TOPOLOGY

Sheet 7, 14.11.2023

Exercise 7.1. Let X be a space, and let $A \subset X$ be a retract of X (in other words, there exists a continuous map $r : X \to A$, so that $A \subset X \xrightarrow{r} A$ is the identity of A).

(a) Let us assume that X satisfies the separation axiom T_1 (the points in X are closed subsets). Let $a \neq b \in X$. Under what conditions is $\{a, b\}$ is a retract of X ?

(b) Show that if X is Hausdorff, then A is a closed subspace of X.

Exercise 7.2. Let (X, x_0) be a pointed space, and $\alpha \in \Omega(X, x_0)$. Let $\bar{\alpha} : (S^1, 1) \to (X, x_0)$ be the only pointed map with $\alpha = \bar{\alpha} \circ q$, where q is the quotient map $[0, 1] \to S^1$, $t \mapsto e^{2\pi i t}$. Furthermore, let $D^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$ be the 2 dimensional disk. Prove that the following statements are equivalent.

(a) We have $\alpha \sim c_{x_0}$ in $\Omega(X, x_0)$, where c_{x_0} is the constant path in X with image x_0.

(b) There exists an extension $h : D^2 \to X$ of $\bar{\alpha} : S^1 \to X$. In other words there exists a map $h : D^2 \to X$, with $\bar{\alpha} = h \circ j$, where $j : S^1 \to D^2$ is the inclusion.

Exercise 7.3. Let $(G, \star)$ be a topological group with e as a neutral element. Prove the following statements.

(a) The operation $\Omega(G, e) \times \Omega(G, e) \to \Omega(G, e)$ defined by $(\alpha, \beta) \mapsto \alpha \star \beta$ with $(\alpha \star \beta)(t) = \big(\alpha(t)\big) \star \big(\beta(t)\big)$, equips $\Omega(G, e)$ with a group structure, having c_e as neutral element.

(b) The operation $\pi_1(G, e) \times \pi_1(G, e) \to \pi_1(G, e)$, $([\alpha], [\beta]) \mapsto [\alpha] \star [\beta] := [\alpha \star \beta]$ is well-defined and produces a group structure on $\pi_1(G, e)$.

(c) We have $[\alpha][\beta] = [\alpha] \star [\beta]$ for all $[\alpha], [\beta] \in \pi_1(G, e)$ (Indication : consider $[\alpha * c_e] \star [c_e * \beta]$, where $*$ is the concatenation of path inducing the usual group structure on π_1).

(d) The group $\pi_1(G, e)$ is Abelian.

Exercise 7.4. Let $B \subset \mathbb{R}^2$ be the union of the circles with center $(0, \frac{1}{n})$ and radius $\frac{1}{n}$ in $\mathbb{R}^2$ for all $n \geq 1$, as studied in Exercise 3.4.(b). Prove the following statements :

(a) There exists a surjective homomorphism $\pi_1(B, 0) \to \prod_{\mathbb{N}} \mathbb{Z}$.

(b) The group $\pi_1(B, 0)$ is not countable.

Exercise 7.5. Let X be a space, and U_1 and U_2 open subsets such that $X = U_1 \cup U_2$, and such that U_1, U_2 and $U_1 \cap U_2$ are path-connected. Let $i : U_1 \cap U_2 \to X$, $i_k : U_1 \cap U_2 \to U_k$ and $j_k : U_k \to X$ be the inclusions ($k = 1, 2$). Let $x_0 \in U_1 \cap U_2$ be chosen. Prove the following variants of the Seifert-Van Kampen theorem.

(a) Let us assume that $i_* : \pi_1(U_1 \cap U_2, x_0) \to \pi_1(X, x_0)$ is trivial. Then j_1 and j_2 induce an isomorphism of groups

$$(\pi_1(U_1, x_0)/N_1) * (\pi_1(U_2, x_0)/N_2) \to \pi_1(X, x_0),$$

where N_k is the normal subgroup of $\pi_1(U_k, x_0)$ generated by $(i_k)_*(\pi_1(U_1 \cap U_2, x_0))$.

(b) Let us assume that $(i_2)_* : \pi_1(U_1 \cap U_2, x_0) \to \pi_1(U_2, x_0)$ is surjective. Then $(j_1)_*$ induces an isomorphism of groups

$$\pi_1(U_1, x_0)/M \to \pi_1(X, x_0),$$

where M is the normal subgroup of $\pi_1(U_1, x_0)$ generated by $(i_1)_*(\mathrm{Ker}(i_2)_*)$.