M1 de Mathématiques Institut Galilée, 2023-2024

GEOMETRY AND TOPOLOGY

Sheet 8, 21.11.2023

Exercise 8.1. Show that $\langle t_1, t_2, \ldots, t_n \mid t_1^2 t_2^2 \ldots t_n^2 \rangle_{\mathrm{ab}} \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}/2$.

Exercise 8.2. Compute $\pi_1(S^n, x)$ for any $n \geq 2$ and a chosen point $x \in S^n$.

Exercise 8.3. Let $n \in \mathbb{N}$. Construct a pointed space (X, x) with $\pi_1(X, x) \cong \mathbb{Z}/n$.

Exercise 8.4. Compute $\pi_1(\mathbb{R}P^n; x)$ for any $n \geq 2$ and a chosen point $x \in \mathbb{R}P^n$.

Exercise 8.5. Let Y be a space. Prove that the following statements are equivalent.

(a) Y is simply-connected (i. e. Y is path-connected, and $\pi_1(Y, y) = 0$ for any $y \in Y$).
(b) Y is non-empty, and for every pair of continuous maps $f, g : S^1 \to Y$, we have that f and g are homotopic.

Theorem (Borsuk-Ulam): *If $f : S^n \to \mathbb{R}^n$ is continuous, then there exists $x \in S^n$ with $f(x) = f(-x)$.*

Exercise 8.6. We prove (among other things) the above theorem for $n = 0, 1, 2$. Let $k \in \mathbb{Z} \setminus \{0\}$, and let $d_k : S^1 \to S^1$ given by $d_k(z) = z^k$ (k-th power of z as a complex number). Let $h : S^1 \to S^1$ be a continuous map with $h(1) = 1$ and $h(-x) = -h(x)$ for all $x \in S^1$.

(a) Show that a continuous map $\bar{h} : S^1 \to S^1$ with $d_2 h = \bar{h} d_2$ exists:

$$\begin{array}{ccc} S^1 & \xrightarrow{h} & S^1 \\ {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle d_2} \\ S^1 & \xrightarrow{\bar{h}} & S^1 . \end{array}$$

(b) Prove that $\bar{h}_* : \pi_*(S^1, 1) \to \pi_*(S^1, 1)$ is injective.
(c) Determine $(d_k)_* : \pi_*(S^1, 1) \to \pi_*(S^1, 1)$.
(d) Prove that $h_* : \pi_*(S^1, 1) \to \pi_*(S^1, 1)$ is injective. In particular, h is not null-homotopic (i.e. not homotopic to a constant map).
(e) Let $f : S^2 \to S^1$ be continuous. Show that an $x \in S^2$ exists with $f(-x) \neq -f(x)$.
Hint: Consider the restriction of f to the equator $S^1 = S^2 \cap (\mathbb{R}^2 \times \{0\})$, and show that it is null-homotopic.
(f) Prove Borsuk-Ulam's theorem for $n = 0, 1$ and 2.
Hint: If $f(x) \neq f(-x)$ holds, then consider $\frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$.

Exercise 8.7. Consider $S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \}$. Let $n, k \in \mathbb{N}$ be positive and relatively prime. Let $\zeta_n = e^{\frac{2\pi i}{n}} \in \mathbb{C}$ and let $h : S^3 \to S^3$ be the continuous map defined by $h(z_1, z_2) = (z_1 \zeta_n, z_2 \zeta_n^k)$.

(a) Prove that h generates a subgroup G of the group of homeomorphisms of S^3, which is cyclic of order n.
(b) Let $L(n, k)$ be the space of orbits S^3/G (with the quotient topology). Compute the fundamental group of of $L(n, k)$.
(c) Show: if $L(n, k)$ and $L(n', k')$ are homotopy equivalent, then $n = n'$.