HOMOTOPY 1

Sheet 1, 06-10 November 2023

Exercise 1.1. Show that the forgetful functor $F : \mathrm{Top}_* \to \mathrm{Top}$ admits the functor $(-)_+ : \mathrm{Top} \to \mathrm{Top}_*$, $X \mapsto X_+ := X \coprod \{*\}$ (adding a disjoint base point) as a left adjoint. Then show that these functors induce an adjunction on associated homotopy categories.

Exercise 1.2. Show that Top admits products and coproducts on arbitrary indexing sets, given by the cartesian product $\prod$ and the disjoint union $\coprod$ (with obvious projections and inclusions), and that these induce products and coproducts on hoTop.

Exercise 1.3. Show that Top_* admits products and coproducts on arbitrary indexing sets, given by the cartesian product (with obvious base-point and projections) and the *wedge-sum*, defined by the quotient $\bigvee_{i\in I}(X_i, x_i) = \left(\coprod_{i\in I}(X_i, x_i)/\coprod_{i\in I} x_i, *\right)$, with $*$ the class of x_i and obvious inclusions. Then, show that these induce products and coproducts on hoTop_*.

Exercise 1.4. Show that the canonical functor $H : \mathrm{Top} \to \mathrm{hoTop}$ satisfies the following universal property: if $F : \mathrm{Top} \to C$ is a functor mapping any homotopy equivalence to an isomorphism, then there exists a unique functor $G : \mathrm{hoTop} \to C$ with $F = GH$:

$$\begin{tikzcd} \mathrm{Top} \arrow[rr, "H"] \arrow[dr, "F"'] & & \mathrm{hoTop} \arrow[dl, dotted, "\exists! G"] \\ & C & \end{tikzcd}$$

Exercise 1.5. Let X be a topological space, and let A, B be closed (resp. open) subspaces of X. Consider the square

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A \cup B \end{tikzcd}$$

where the maps are the inclusions. Is this square cartesian (resp. cocartesian) in Top?

Exercise 1.6. Let $n \in \mathbb{N}$, $n \geq 1$, and define the sphere S^{n-1} and the disk D^n as the subspaces

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}, \quad \text{and} \quad D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}\,.$$

(a) Prove that the sphere S^n is homeomorphic to the push-out of $D^n \leftarrow S^{n-1} \to D^n$ (where the maps are the inclusions), so that there exists a cocartesian square

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd}$$

in the category Top.

(b) Assuming that the identity of S^n is not homotopic to a constant map (which is the case, for example by an argument using homology), explain why this square is not a cocartesian square in hoTop.

Exercise 1.7. Consider the Sierpinski space Z with two points x,y, and with topology given by the family of subsets $\{\emptyset, \{x\}, Z\}$. Is Z contractible ?

Exercise 1.8. Let $\mathbb{R}$ and $\mathbb{R}^2$ be endowed with the Euclidean topology. We define topological spaces A, B, C and D as certain "unions of cirlces", as follows:

(a) A is the union of the circles with center $(0, n)$ and radius n in $\mathbb{R}^2$ for all $n \geq 1$, with the subspace topology;
(b) B is the union of the circles with center $(0, \frac{1}{n})$ and radius $\frac{1}{n}$ in $\mathbb{R}^2$ for all $n \geq 1$, with the subspace topology (the so-called Hawaiian earrings);
(c) C is the quotient space $\mathbb{R}/\sim$, where $x \sim y$ exactly if $x = y$ or $x, y \in \mathbb{Z}$. In other words, C is the quotient of $\mathbb{R}$ by its subspace $\mathbb{Z}$ (not the quotient as groups!).
(d) $D = \bigvee_{\mathbb{Z}} S^1$, i.e. D is the underlying space of the coprduct in Top_* of copies of S^1 indexed by $\mathbb{Z}$, where the unit circle $S^1 \subset \mathbb{C}$ has 1 as basepoint.

Discuss which of these spaces are homeomorphic and which are homotopy equivalent.

Exercise 1.9. Let $(X, A, B, *)$ be a pointed triple, i.e. $* \in B \subset A \subset X$. Prove that the sequence

$$\cdots \to \pi_{n+1}(X, A, *) \xrightarrow{d_{n+1}} \pi_n(A, B, *) \xrightarrow{i_*} \pi_n(X, B, *) \xrightarrow{j_*} \pi_n(X, A, *) \to \cdots \to \pi_1(X, A, *)$$

is exact. Here i_* and j_* are induced by the inclusions, and d_{n+1} is the composition

$$\pi_{n+1}(X, A, *) \xrightarrow{\partial_{n+1}} \pi_n(A, *) \to \pi_n(A, B, *).$$

Exercise 1.10. Let $X_0 = \{*\} \subset X_1 \subset X_2 \subset X_3 \subset \cdots$ be a sequence of inclusions of Hausdorff spaces, and consider its colimit $X = \bigcup_{n \geq 0} X_n$. Prove that the morphisms $\pi_*(X_n, *) \to \pi_*(X, *)$ induced by the inclusion produce an isomorphism

$$\mathrm{colim}_n\, \pi_*(X_n, *) \to \pi_*(X, *)\,.$$