HOMOTOPY 1

Sheet 2, 13-17.11.2023

Exercise 2.1. Can you compute $\pi_1(X, x_0)$, where X is one of the "union of circles" given in Exercise 1.8, with x_0 the point common to all circles ?

Exercise 2.2. Show that if the map $f : X \to Y$ in Top is a homotopy equivalence, then $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $x \in X$ and all $n \in \mathbb{N}$.

Exercise 2.3. Prove that the map $p : \mathbb{C} \to \mathbb{C}$, $p(z) = z^2$ has the homotopy lifting property for $I^0 = \{0\}$ but not for I^1.

Exercise 2.4. Prove the following assertions.

(a) The composition of two fibrations is a fibration.
(b) The product of two fibrations is a fibration.
(c) If $p : E \to B$ is a fibration with B path-connected and $E \neq \emptyset$, then p is surjective.
(d) Assume given a pull-back square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ Y \arrow[r, "g"] & B \end{tikzcd}$$

where p is a fibration. Prove that q is also a fibration. In the pointed case, prove that the restriction of f induces a homeomorphism $Z \to F$, where Z and F are the fibres of q and p, respectively.

Exercise 2.5. Consider the triangle $E \subset \mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Let $B = [0, 1]$, and let $p : E \to B$ be the restriction of the projection $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$. Prove the following assertions.

(a) The map p is a fibration.
(b) The map p is not a fibre bundle.

Exercise 2.6. Consider $n \geq 1$.

(a) Compute $\pi_m(\mathbb{R}P^n, *)$ and $\pi_\ell(\mathbb{C}P^n, *)$ for as many values of m and ℓ as possible.
(b) Using exercise 1.10, compute $\pi_*(\mathbb{R}P^\infty, *)$ and $\pi_*(\mathbb{C}P^\infty, *)$, where $\mathbb{F}P^\infty = \mathrm{Colim}_n\, \mathbb{F}P^n$.

Exercise 2.7. Prove (some of) the properties of the compact-open topology of mapping spaces listed in Lemma 1.35.

Exercise 2.8. Let $(X, x_0) \in \mathrm{Top}_*$. Prove that the internal binary product on $(\Omega(X, x_0), *)$ given by the concatenation of loops endows $(\Omega(X, x_0), *)$ with the structure of a group in hoTop_*. Proved that this induces a group structure on $\pi_n(\Omega(X, x_0), *)$ for all $n \geq 0$, that coincides with the group structure of homotopy groups for $n \geq 1$.