HOMOTOPY 1

Sheet 3, 20.11.2023

Exercise 3.1. Consider

$$E = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1,\ y = x-1 \text{ or } y = 1/n, n \in \mathbb{N} \setminus \{0\}\} \subset \mathbb{R}^2,$$

and define $p : E \to B = [0,1]$ by $p(x,y) = x$. Prove that p is a Serre fibration, but is not a Hurewicz fibration.

Exercise 3.2. Let (X, A) be a pair of spaces and $(Y, *)$ be a pointed space.

(a) Let $q : X \to X/A$ be the quotient map, and take $\{A\}$ as base point of X/A. Prove that the map

$$\bar{q} : \mathrm{map}_*(X/A, Y) \to \mathrm{map}\big((X, A), (Y, *)\big), \quad f \mapsto fq$$

is continuous and bijective. Here $\mathrm{map}_*(X/A, Y)$ denotes the subspace of $\mathrm{map}(X/A, Y)$ consisting of pointed maps. Deduce that there is a bijection in hoTop_*

$$\bar{q} : [X/A, Y]_* \to [(X, A), (Y, *)]\,.$$

(b) What about the case $A = \emptyset$?

(c) Deduce from (a) that there is a natural bijection

$$[S^n, X]_* \to \pi_n(X, *) \quad \text{and} \quad [(D^n, S^{n-1}), (X, A)]_* \to \pi_n(X, A, *)$$

where $[-,-]_*$ denotes the set of pointed homotopy classes.

Exercise 3.3. Denote by $\vee$ the coproduct on Top_* (describe it!). It is also a coproduct on hoTop. Define the notion of a co-H-space (the categorical dual of an H-space). If $(X, *)$ is a co-H-space, then $[(X, *), (Y, *)]_*$ has an associative and unital product. Prove that for $n \geq 1$, S^n admits a co-H-group structure, such that the induced product on $[(S^n, *), (X, *)]_*$ is the same as the product on $\pi_n(X, *)$.
Why do co-groups not show up in any algebra lecture ?

Exercise 3.4. Let $p : (E, e) \to (B, b)$ be a Serre fibration with fibre $p^{-1}(b) = F$. Prove that there is an induced map

$$\Omega^n(p) : (\Omega^n(E, e), *) \to (\Omega^n(B, b), *)\,,$$

and that it is a Serre fibration with fibre $\Omega^n(F, e)$.

Exercise 3.5. Let $(X, *)$ be a pointed space, and set $\mathbb{R}_+ = \{t \in \mathbb{R} \,|\, t \geq 0\}$. Define the *space of Moore loops* as

$$\Omega^M(X, *) = \{(t, w) \in \mathbb{R}_+ \times F(\mathbb{R}_+, X) \,|\, w(0) = *,\ \text{and } w(s) = * \text{ if } s \geq t\},$$

with the subspace topology of $\mathbb{R}_+ \times F(\mathbb{R}_+, X)$. Define a product

$$\mu : \Omega^M(X, *) \times \Omega^M(X, *) \to \Omega^M(X, *), \quad \mu\big((s, v), (t, w)\big) = (s + t, u),$$

where u is the obvious concatenation of v and w. We take $* = (0, c_*)$ as base point of $\Omega^M(X, *)$. Prove the following assertions.

(a) $\Omega^M(X,*)$ is a topological monoid.
(b) The (unpointed) inclusion $\Omega(X,*) \subset \Omega^M(X,*)$ is a deformation retract. The retraction can be chosen pointed, and the products are compatible up to homotopy.

Exercise 3.6. Let $k \in \mathbb{N}$, $k \geq 1$ be chosen, and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $n \geq k$, consider the inclusions

$$V^{\mathbb{F}}_{n,k} \subset V^{\mathbb{F}}_{n+1,k} \quad \text{and} \quad G^{\mathbb{F}}_{n,k} \subset G^{\mathbb{F}}_{n+1,k}$$

induced by the inclusion $\mathbb{F}^n = \mathbb{F}^n \times \{0\} \subset \mathbb{F}^{n+1}$. Define $V^{\mathbb{F}}_{\infty,k} = \operatorname{colim}_n V^{\mathbb{F}}_{n,k}$ and $G^{\mathbb{F}}_{\infty,k} = \operatorname{colim}_n G^{\mathbb{F}}_{n,k}$.

(a) Prove that the induced map $V^{\mathbb{F}}_{\infty,k} \to G^{\mathbb{F}}_{\infty,k}$ is a Serre fibration.
(b) Prove that for $n \geq 1$ there is an isomorphism of groups

$$\pi_{n+1}(G^{\mathbb{F}}_{\infty,k}, *) \to \pi_n(G(k), 1)$$

where $G(k) = O(k)$, $U(k)$ for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ respectively.
(c) Compute $\pi_n(G^{\mathbb{R}}_{\infty,k}, *)$ for $n \leq 1$ and $\pi_m(G^{\mathbb{C}}_{\infty,k}, *)$ for $m \leq 2$.

Exercise 3.7. Let $p : E \to B$ be a Hurewicz fibration.

(a) Assume w is a path in B, with $a = w(0)$ and $b = w(1)$. Using the homotopy lifting property of p, define a map $h_w : F_a \to F_b$ such that that $[h_w]$ depends only on $[w]$.
(b) Prove the existence of a functor $F^p : \Pi(B) \to \text{hoTop}$ with $F^p(a) = F_a$. Deduce that if a, b lie in the same path component of B, then $F_a \simeq F_b$.

Exercise 3.8. Prove the following variant of Lemma 1.45 of the lecture: Let (X, A) be a pair and $n \geq 1$. The following conditions are equivalent:

(a) The pair (X, A) is n-connected;
(b) Any map of pairs $f : (I^n, \partial I^n) \to (X, A)$ is homotopic (as maps of pairs) to a map $g : (I^n, \partial I^n) \to (X, A)$ with $g(I^n) \subset A$.

Exercise 3.9. Prove that $\mathbb{R}P^n$ is simple if and only if n is odd.

Exercise 3.10. Let $(X, *)$ be a pointed space, and identify $\pi_n(X, *)$ with $[(S^n, *), (X, *)]_*$. Define the *composition product* as

$$\pi_p(X, *) \times \pi_q(S^p, *) \xrightarrow{\circ} \pi_q(X), \quad ([f], [\alpha]) \mapsto [f] \circ [\alpha] := [f\alpha].$$

Prove the following assertions.

(a) The composition product is well-defined, natural in X and associtive in the sense that $([f] \circ [\alpha]) \circ [\beta] = [f] \circ ([\alpha] \circ [\beta])$ in $\pi_r(X, *)$, for $\beta \in \pi_r(S^q, *)$.
(b) It is additive in the second variable: for any $[\alpha_1], [\alpha_2] \in \pi_q(S^p, *)$, we have

$$[f] \circ ([\alpha_1] + [\alpha_2]) = [f] \circ [\alpha_1] + [f] \circ [\alpha_2].$$

(c) Let $h \in \pi_3(S^2, *)$ be the generator given by the Hopf-map, and $1 = [\mathrm{id}_{S^2}] \in \pi_2(S^2, *)$. Show that $(-1) \circ h = h \in \pi_3(S^2, *)$. This shows that $- \circ [\alpha]$ is not necessarily additive.
(d) Let $\beta \in \pi_{q-1}(S^{p-1}, *)$. Show that $- \circ \Sigma_*[\beta] : \pi_p(X, *) \to \pi_q(X, *)$ is additive.
(e) Prove that $2h$ is in the kernel of $\Sigma_* : \pi_3(S^2, *) \to \pi_4(S^3, *)$. Deduce that the group $\pi_4(S^3, *) \cong \pi_1^s$ has order 1 or 2 (we will prove later that the order is 2).