HOMOTOPY 1

Sheet 4, 27.11.2023

Exercise 4.1. Which of the "union of cercles" of Exercise 1.8 admits a structure of a *CW*-complex ?

Exercise 4.2. Let $p : E \to B$ be a Serre fibration, $e \in E$ and $b = p(e)$. Let $i : (F, e) \to (E, e)$ be the inclusion of the fiber. Let $n \geq 0$. Prove the following statements.

(a) If i is homotopic relative $\{e\}$ to the constant mapping, then we have a short exact sequence

$$0 \to \pi_{n+1}(E, e) \xrightarrow{p_*} \pi_{n+1}(B, b) \xrightarrow{\partial} \pi_n(F, e) \to 0,$$

and ∂ has a section.

(b) We have isomorphisms

$$\pi_n(S^4, *) \cong \pi_n(S^7, *) \oplus \pi_{n-1}(S^3, *).$$

In particular. $\pi_7(S^4, *) \cong \mathbb{Z} \oplus \pi_6(S^3, *)$ holds.

Exercise 4.3. Let $(X, *)$ and $(Y, *)$ be pointed *CW*-complexes with X k-connected and Y ℓ-connected. We assume that X or Y is locally-compact. Prove the following statements.

(a) The pair $(X \times Y, X \vee Y)$ is $(k+\ell+1)$-connected. In particular $X \wedge Y$ is also $(k+\ell+1)$-connected.

(b) The inclusions induce an isomorphism $\pi_n(X, *) \oplus \pi_n(Y, *) \to \pi_n(X \vee Y, *)$ for all $1 \leq n \leq k + \ell$.

(c) Let A be a set and $n \geq 2$. The inclusions induce an Isomorphism

$$\bigoplus_{a \in A} \pi_n(S^n) \to \pi_n(\bigvee_{a \in A} S^n).$$

Exercise 4.4. Define the Euler characteristic $\chi(X)$ of a finite CW-complex X to be the alternating sum $\sum_{i=0}^{\infty}(-1)^n \gamma_n(X)$, where $\gamma_n(X)$ is the number of n-cells of X. Let A be a subcomplex of a CW-complex X, let Y be a CW-complex, let $f : A \to Y$ be a cellular map, and let $Y \cup_f X$ be the push-out of f and the inclusion $A \to X$.

(a) Show that $Y \cup_f X$ is a CW complex with Y as a sub-complex and X/A as a quotient complex. Formulate and prove a formula relating the Euler characteristics A, X, Y, and $Y \cup_f X$ when X and Y are finite.

(b) Prove that $\chi(X)$ depends only on the homotopy type of X, not on its decomposition as a finite CW-complex.

Exercise 4.5. Let X and Y be countable CW-complexes. Then $X \times Y$ is a CW-complex with the (ordinary) product topology (with cells given by the product-cells).

Exercise 4.6. Let X be a CW-complex

(a) Show that each neighbourhood U of a point x of a CW-complex X contains a neighbourhood V which is pointed contractible to x. Thus X has a universal covering.
(b) Show that the universal covering of X has a CW-structure, with a cell decomposition such that its automorphism group permutes the cells freely.

Exercise 4.7. Show that the CW-complexes $S^3 \times \mathbb{C}P^\infty$ and S^2 have isomorphic homotopy groups but are not homotopy equivalent. Show that the same statement holds for $S^n \times \mathbb{R}P^m$ and $S^m \times \mathbb{R}P^n$ for suitable values of m and n (and these are finite CW-complexes!).