HOMOTOPY 1

Sheet 5, 04.12.2023

Exercise 5.1. We want to show weak homotopy invariance of singular homology. Let $(S_*(X), d_*)$ denote the singular chain complex of a space X. If $k, n \in \mathbb{N}$, view the standard n-simplex as a CW-complex with $\binom{k+1}{n+1}$ n-cells given by its n-dimensional faces, and let $\Delta^{k,n}$ be its n-skeleton. For $(X, A) \in \mathrm{Top}^2$, let $\mathrm{Sing}_k^{(A,n)}(X) = \mathrm{Top}^2\big((\Delta^k, \Delta^{k,n}), (X, A)\big)$, and $S_k^{(A,n)}(X) = \mathbb{Z}\{\mathrm{Sing}_k^{(A,n)}(X)\} \subset S_k(X)$.

(a) Show that this defines the inclusion of a sub-chain-complex $\phi : (S_*^{(A,n)}(X), d_*) \subset (S_*(X), d_*)$.

Suppose now that (X, A) is a n-connected pair.

(b) By induction on k, construct for any k and any k-simplex $\sigma_0 : \Delta^k \to X$ a homotopy $H^\sigma : \Delta^k \times I \to X$ between σ_0 and $\sigma_1 = H_1^\sigma$, such that:
   (1) $\sigma_1 \in \mathrm{Sing}_k^{(A,n)}(X)$
   (2) $H_t^\sigma = \sigma_0$ if $\sigma \in \mathrm{Sing}_k^{(A,n)}(X)$, for any $t \in I$.
   (3) For any face $d_i : \Delta^{k-1} \to \Delta^k$, we have $H^\sigma \circ (d_i \times \mathrm{id}) = H^{\sigma \circ d_i}$.

(c) Show that the rule $\sigma_0 \mapsto \sigma_1$ defines a morphism of chain complexes $\psi : (S_*(X), d_*) \to (S_*^{(A,n)}(X), d_*)$ with $\psi \circ \phi = \mathrm{id}$.

(d) Construct a chain homotopy between $\phi \circ \psi$ and id, using, for any $\sigma \in \mathrm{Sing}_k(X)$, the composition
$$H_*^\sigma \circ h : S_k(\Delta^k) \to S_{k+1}(\Delta^k \times I) \to S_{k+1}(X),$$
where h is the standard chain homotopy (used in proving homotopy invariance of homology) between the chain maps induced by the inclusion of the bottom face and top face of the cylinder $\Delta^k \times I$.

(e) Deduce that the homomorphism $H_k(A; \mathbb{Z}) \to H_k(X; \mathbb{Z})$ induced by the inclusion $A \to X$ is an isomorphism for $k \leq n$.

In the next three exercises, we construct and study the Poincaré sphere.

Exercise 5.2. Prove that there is a group homorphism $\phi : S^3 \to SO(3)$ (the rotation group of $\mathbb{R}^3$) realizing S^3 as the universal cover of $SO(3)$, and $SO(3)$ as the quotient of S^3 by the subroup $\mathbb{Z}/2$.

Hint: Show that $SU(2)$, the group of unitary 2×2-matrices of determinant 1, is isomorphic to S^3, the group of unit quaternions. To construct the group homomorphism $SU(2) \to SO(3)$, consider the action of S^3 on $\mathbb{H}$ by conjugation. This representation splits as the trivial representation on $\mathbb{R}\{1\}$, and a representation on $\mathbb{R}\{i, j, k\}$ (by rotations, which we see easily from identifictation of S^3 with $SU(2)$!).

Exercise 5.3. We study the finite subgroup $I < SO(3)$ of rotations preserving an icosahedra $X \subset \mathbb{R}^3$ centered at the origin (i.e. mapping it to itself), called the *icosahedral group.*

(a) Let $I_n = \{f \in I\ ;\ f \text{ has exact order } n\}$. Show that $I = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_5$ (the identity, and the rotations with axes cutting X along the middle of an edge, of a face, or a vertex, respectively). Deduce that I has order 60.

(b) Show that I is simple (its only normal subgroups N are $\{\mathrm{id}\}$ and I). To do so, first show that the cyclic subgroups of order 2 (axe through the middle of an edge) are conjugated to each other; similarly for those or order 3 or 5. Deduce that a normal subgroup is of order $1 + a\mathrm{Card}(I_2) + 2b\mathrm{Card}(I_3) + 4c\mathrm{Card}(I_5)$ for some $a, b, c \in \{0, 1\}$; conclude using that this number must divide 60.

(c) Deduce that I is perfect (i.e. $I = [I, I]$, or $I_{\mathrm{ab}} = 0$).

Exercise 5.4. The *binary icosahedral group* is the subgroup $I' = \phi^{-1}(I)$ of S^3, of order 120.

(a) Show that $-1 \in [I', I']$, and deduce that I' is perfect. To do so, assume that the icosahedron X is in a position in $\mathbb{R}\{i, j, k\}$ so that the quaternions i and j belong to I', and use $-1 = [i, j]$.

(b) The quotient $P = S^3/I'$ is called the Poincaré sphere. Explain why P is a smooth, compact, oriented manifold of dimension 3.

(c) Using the Hurewicz theorem for π_1 and Poincaré duality, show that the quotient map $S^3 \to P$ induces an isomorphism in homology.

(d) Using the Serre fibration $I' \to S^3 \to P$, explain how the homotopy groups of S^3 and P agree or differ.

Exercise 5.5. Check that the pair $(|\ |, \mathrm{Sing})$ is an adjoint pair

$$|\ | : \widehat{\Delta} \longleftrightarrow \mathrm{Top} : \mathrm{Sing}\,.$$

Exercise 5.6. Let C and D be small categories, $F, G : C \to D$ functors and $\eta : F \Rightarrow G$ a natural transformation. Show that η corresponds to a functor $C \times [1] \to D$, and induces a homotopy between the maps $|NF|$ and $|NG|$.

Exercise 5.7. Show that if a small category C has an initial or terminal object, then its classifying space is contractible.

Exercise 5.8. Describe explicitly $B\mathbb{Z}/2$, the classifying space of the group $\mathbb{Z}/2$ (viewed as a category), as a CW-complex.

Exercise 5.9. Determine a finite category C_n such that the geometric realization of NC_n is the sphere S^n, up to homeomorphism.

Exercise 5.10. Show that the nerve functor N from small categories to simplicial sets is right adjoined to the *path functor* P from simplicial sets to small categories, that associates to a simplical set X the small category whose set of objects is the set X_0, with morphisms of the form $x : d_1x \to d_0x$ for all $x \in X_1$, with relations of the form $d_1y = d_0y \circ d_2y$ for all $y \in X_2$.

Exercise 5.11. Let X be a simplicial set. Under what conditions on X does X determine a category C such that $X = NC$ (the nerve of C) ?