HOMOTOPY 1

Sheet 6, 11.12.2022

Exercise 6.1. Show that a map f in Top is a Serre fibration if and only if $\mathrm{Sing}(f)$ in $\widehat{\Delta}$ is a Kan fibration. Define the notion of the *fibre* of a kan fibration $f : X \to Y$ above $y \in Y_0$ as a simplicial sub-set of X, and show that it is a Kan complex.

Exercise 6.2. Show that NC is a Kan complex if and only if C is a groupoïd, i.e. a small category where all morphisms are isomorphisms. Deduce that if $n \geq 1$, Δ^n is not a Kan complex. Show that if X is a Kan complex, then the path category $\tau(X)$ is a groupoïd.

Exercise 6.3. If G is a group, let $\underline{G}$ be the small category with set of objects equal to the underlying set of G, and exactly one morphism $g \to h$, denoted $g^{-1}h$, for any pair of objects (g, h) in G.

(a) Show that $|N\underline{G}|$ is contractible.
(b) Show that G acts freely on $N\underline{G}$ and that we have a (quotient) map of simplicial sets $q : N\underline{G} \to (N\underline{G})/G$
(c) Show that q is kan fibration, and describe its fiber.
(d) Admitting the that the geometric realzation of a Kan fibration is a Serre fibration, compute the homotopy groups of $BG = |NG|$.

Exercise 6.4. Show that the underlying simplicial set of a simplicial group (a functor $\Delta^{\mathrm{op}} \to$ Groups) is a Kan complex.

Exercise 6.5. Let X be an ∞-category.

(a) Define the (non-unique) composition of morphisms in X with the help of the inner horn-extensions.
(b) Say that $f, g : x \to y$ are homotopic if there is a 2-simplex whose boundary is given by the 1-simplices $1 : x \to x$, $f : x \to y$ and $g : x \to y$. Show that this is an equivalence relation.
(c) Show that there exists an ordinary category $\mathrm{ho}(X)$, with same objects as X, and morphisms are homotopy classes of maps, where composition of two morphisms is given by the homotopy class of any choice of compostion. This is the *homotopy category* of X.
(d) Show that there is a natural isomorphism of categories $\mathrm{ho}(X) \cong \tau(X)$.