EXERCISES

Sheet 1, 11.09.2019

Exercise 1.1. Let G and H be two groups, considered as categories with one object, which we denote as $\mathbf{G}$ and $\mathbf{H}$.

(a) What are the functors $F : \mathbf{G} \to \mathbf{H}$?
(b) Given two functors $F_1, F_2 : \mathbf{G} \to \mathbf{H}$, under which conditions does a natural transformation $\eta : F_1 \to F_2$ exist ?

Exercise 1.2. Let **Grp** be the category of groups and group homomorphisms, and let **Ab** be the full subcategory of abelian groups. If G is a group, let $[G, G]$ be the subgroup of G generated by the commutators $[g, h] = ghg^{-1}h^{-1}$ for all $g, h \in G$. Prove the following assertions:

(a) $[G, G]$ is a normal subgroup of G, and $G^{\mathrm{ab}} = G/[G, G]$ is an abelian group.
(b) The correspondence $G \mapsto G^{\mathrm{ab}}$ is part of a functor $(-)^{\mathrm{ab}} : \mathbf{Grp} \to \mathbf{Ab}$ such that the composite functor
$$\mathbf{Ab} \overset{\subseteq}{\to} \mathbf{Grp} \xrightarrow{(-)^{\mathrm{ab}}} \mathbf{Ab}$$
is the identity.
(c) Prove that the functor $(-)^{\mathrm{ab}}$ is a left adjoint functor.

Exercise 1.3. Let G be a group and let
$$\mathrm{Z}(G) = \{g \in G \mid hg = gh \text{ for all } h \in G\}$$
be the center of G. Obviously, $\mathrm{Z}(G)$ is an abelian subgroup of G. Prove that there exists no functor $F : \mathbf{Grp} \to \mathbf{Ab}$ which on objects is given by $F(G) = \mathrm{Z}(G)$.

Exercise 1.4. In a category $\mathcal{C}$, a morphism $f \in \mathcal{C}(X, Y)$ is called a *epimorphism* if it is *right cancellable*. This means that for any object W in $\mathcal{C}$ and any pair of morphisms $g, h \in \mathcal{C}(Y, W)$, we have the implication
$$(gf = hf) \Longrightarrow (g = h)\,.$$

(1) Define the dual notion of a *monomorphism*.
(2) Prove that in the category of sets, a monomorphism is an injection, and an epimorphism is a surjection.
(3) Let **Ring** be the category of unital rings and ring homomorphisms. Consider the inclusion $f : \mathbb{Z} \to \mathbb{Q}$.
   (a) Is f a monomorphism in **Ring**?
   (b) Is f an epimorphism in **Ring**?
   (c) Is f an isomorphism in **Ring**?

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Exercise 1.5. Let $(P, \leq)$ and $(Q, \leq)$ be two partially ordered sets, viewed as small categories $\mathbf{P}$ and $\mathbf{Q}$.

(a) Describe the functors $\mathbf{P} \to \mathbf{Q}$.
(b) Given two functors $F, G : \mathbf{P} \to \mathbf{Q}$, under which conditions does a natural transformation $F \Longrightarrow G$ exist? If so, how many such transformations exist ?

Exercise 1.6. Let $\mathbf{C}$ be a category, and for $X \in \mathrm{Ob}(\mathbf{C})$, let $\varphi^X = \mathbf{C}(X, -) : \mathbf{C} \to \mathbf{Set}$ be the functor coreprensented by X. Let $Y \in \mathrm{Ob}(\mathbf{C})$, and let $\eta : \varphi^X \Longrightarrow \varphi^Y$ be a natural transformation.

(a) Prove that there exists a unique $f \in \mathbf{C}(Y, X)$ such that $\eta = \varphi^f$, where φ^f is defined by $\varphi^f_Z : \mathbf{C}(X, Z) \to \mathbf{C}(Y, Z)$, $g \mapsto g \circ f$.
(b) Formulate and prove the corresponding dual statement for represented cofunctors.

Exercise 1.7. Prove that a category that admits all coproducts and co-equalizers is cocomplete. Use this to prove that the category of modules over a commutative ring is cocomplete.

Exercise 1.8. Explicitly describe push-outs in the following categories:

(a) The category of sets;
(b) The category of abelian groups;
(c) The category of commutative (unital) rings;
(d) The category of unital rings.