EXERCISES

Sheet 2, 18.09.2019

Exercise 2.1. Give a proof of the Yoneda Lemma stated in the lecture. Explain why the Yoneda lemma implies, for example, that when a colimit exists, it is unique up to isomorphism.

Exercise 2.2. Let (L, R) be an adjoint pair of functors $L : \mathbf{C} \to \mathbf{D}$, $R : \mathbf{D} \to \mathbf{C}$. Prove the following properties:

(a) L preserves colimits.
(b) R preserves limits.

Hint: For (a), prove and use that if I is a small category and $F \in C^I$, then for any object $Y \in C$, we have an isomorphism

$$C(\operatorname{colim}_I F, Y) \cong \lim_I C(F(-), Y)\,.$$

Then apply Yoneda.

Exercise 2.3. Let $L : C \to D$ be a functor. Prove that if L admits a right adjoint, it is unique up to natural isomorphism.

Exercise 2.4. An object c_0 in a category C is called *initial* if for any object x in C, $C(c_0, x)$ has a unique element. An object d_0 in C is called *terminal* if it is initial in C^{op}.

(a) Prove that if it exists, an initial (terminal) object is unique up to isomorphism.
(b) Do the following categories admit initial or terminal objects ?

$$\mathbf{Sets},\ \mathbf{Top},\ \mathbf{Top}_*,\ \mathbf{Grp},\ \mathbf{Ring},\ \text{ and } \mathbf{G} \text{ (associated to a group } G).$$

(c) Prove that if I is a small category admitting an initial object i_0, and if $F \in C^I$, then $F(i_0)$ is a limit of F (with the obvious natural transformation $\Delta_{F(i_0)} \Longrightarrow F$). What is the dual statement ?

Exercise 2.5. Let $n \in \mathbb{N}$, $n \geq 1$, and define the sphere S^{n-1} and the disk D^n as the subspaces

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}, \quad \text{and} \quad D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}\,.$$

(a) Prove that the sphere S^n is homeomorphic to the push-out of $D^n \leftarrow S^{n-1} \to D^n$ (where the maps are the inclusions), so that there exists a cocartesian square

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd}$$

in the category of topological spaces.

(b) Assuming that the identity of S^n is not homotopic to a constant map (which is the case), explain why this square is not cocartesian in the homotopy category.

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http://www.math.univ-paris13.fr/~ausoni/m2-2019.html

Exercise 2.6. Let X be a topological space, and let A, B be closed (resp. open) subspaces of X. Consider the square

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A \cup B \end{tikzcd}$$

where the maps are the inclusions. Is this square Cartesian ? Cocartesian ?

Exercise 2.7. Consider the Sierpinski space Z with two points x,y, and with topology given by the family of subsets $\{\emptyset, \{x\}, X\}$. Is Z contractible ?

Exercise 2.8. Consider the following subspaces of $\mathbb{R}$:

$$\mathbb{Z}, \quad X = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \dots\}, \quad Y = X \cup \{0\}\,.$$

Which have the same homotopy type ?