EXERCISES

Sheet 3, 24.09.2019

Exercise 3.1. Let $(G, \star)$ be a topological group with neutral element e. Let c_e be the constant path in G with image $\{e\}$. Denote the set of loops in G based at e by $\Omega(G, e)$. Prove the following assertions:

(a) The operation $\Omega(G, e) \times \Omega(G, e) \to \Omega(G, e)$, given by $(\alpha, \beta) \mapsto \alpha * \beta$ with $(\alpha * \beta)(t) = \big(\alpha(t)\big) \star \big(\beta(t)\big)$ for all $t \in [0, 1]$ defines a group structure on $\Omega(G, e)$ with c_e as neutral element.

(b) The operation $\pi_1(G, e) \times \pi_1(G, e) \to \pi_1(G, e)$, $([\alpha], [\beta]) \mapsto [\alpha] * [\beta] := [\alpha * \beta]$ is well-defined and is a group structure on $\pi_1(G, e)$.

(c) We have $[\alpha][\beta] = [\alpha] * [\beta]$ for all $[\alpha], [\beta] \in \pi_1(G, e)$ (Indication: consider $[\alpha c_e] * [c_e \beta]$).

(d) The group $\pi_1(G, e)$ is abelian.

Exercise 3.2. Consider the *Hawaiian ear-rings*, defined as the subspace B of $\mathbb{R}^2$ that is the union of the circle $\bigcup_n C_n$, where C_n is the circle of center $(0, \frac{1}{n})$ and radius $\frac{1}{n}$ in $\mathbb{R}^2$, for all $n \geq 1$. Prove the existence of a surjective homomorphism

$$\pi_1(B, 0) \to \prod_{\mathbb{N}} \mathbb{Z}\,.$$

Deduce that $\pi_1(B, 0)$ is uncountable.

Exercise 3.3. Prouve that the functor F from the category of groups to the category of groupoids (which sends a group G to the groupoid with one object and G as morphisms) preserves push-outs.

Exercise 3.4. Let X be a space, U_1 and U_2 open subspaces with $X = U_1 \cup U_2$, and such that U_1, U_2 and $U_1 \cap U_2$ are path-connected. Let $i : U_1 \cap U_2 \to X$, $i_k : U_1 \cap U_2 \to U_k$ and $j_k : U_k \to X$ be the inclusions $(k = 1, 2)$. Choose $x_0 \in U_1 \cap U_2$. Prove the following variants of the Seifert-Van Kampen Theorem.

(a) Assume $i_* : \pi_1(U_1 \cap U_2, x_0) \to \pi_1(X, x_0)$ is trivial. Then j_1 and j_2 induce a group isomorphism

$$(\pi_1(U_1, x_0)/N_1) * (\pi_1(U_2, x_0)/N_2) \to \pi_1(X, x_0)\,,$$

where N_k is the normal subgroup of $\pi_1(U_k, x_0)$ generated by $(i_k)_*(\pi_1(U_1 \cap U_2, x_0))$.

(b) Assume $(i_2)_* : \pi_1(U_1 \cap U_2, x_0) \to \pi_1(U_2, x_0)$ is surjective. Then $(j_1)_*$ induces an Isomorphism of groups

$$\pi_1(U_1, x_0)/M \to \pi_1(X, x_0)\,,$$

where M is the normal sub-group of $\pi_1(U_1, x_0)$ generated by $(i_1)_*(\mathrm{Kern}(i_2)_*)$.

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http://www.math.univ-paris13.fr/∼ausoni/m2-2019.html

Exercise 3.5. Prove with the Theorem of Seifert-Van Kampen that $\pi_1(S^1 \times S^1, (1,1))$ and $\mathbb{Z} \times \mathbb{Z}$ are isomorphic. Assuming $\pi_1(S^1, 1) \cong \mathbb{Z}$, do we really need Seifert-Van Kampen?

Exercise 3.6. Use the Theorem of Seifert-Van Kampen, to compute $\pi_1(\mathbb{R}P^2, *)$, where $\mathbb{R}P^2$ is the real projective plane.

Exercise 3.7. Consider the following circles in $\mathbb{R}^3$: C_1 is in the $z = 0$ plane, centered at 0 of radius 1; C_2 is in the $z = 0$ plane, centered at $(3, 0, 0)$ of radius 1, and C_3 is in the $y = 0$ plane, centered at $(\frac{3}{2}, 0, 0)$ of radius 1. Note that C_1 and C_3 are linked. Compute the fundamental group of $\mathbb{R}^3 \setminus (C_1 \cup C_2)$ and $\mathbb{R}^3 \setminus (C_1 \cup C_3)$.

Exercise 3.8. Prove that the category of pointed spaces Top_* admits push-outs. We denote by $X \vee Y$ the coproduct of pointed spaces. Prove that there is a natural map $X \vee Y \to X \times Y$ that restricts to a homeomorphism of $X \vee Y$ onto the subspace

$$W = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subset X \times Y\,,$$

where $x_0 \in X$ and $y_0 \in Y$ are the base-points. Does $\vee$ induce a coproduct on hoTop_* ?

Exercise 3.9. For $n \in \mathbb{N}$, let $S^n = \{x \in \mathbb{R}^{n+1}\,;\, \|x\| = 1\}$ be the n-dimensional sphere, and let $D^{n+1} = \{x \in \mathbb{R}^{n+1}\,;\, \|x\| \leq 1\}$ be the $(n+1)$-dimensional disk.

(a) Prove that for $n \geq 1$, S^n admits the structure of a *co-group* in hoTop_*, a structure enhancing $[(S^n, *), (X, x_0)]_*$ with a natural group structure. Prove that we have a natural bijection

$$\pi_n(X, x_0) \cong [(S^n, *), (X, x_0)]_*$$

for $n \geq 0$ that is a natural group isomorphism for $n \geq 1$.

(b) Similarly, prove that we have a natural group isomorphism

$$\pi_{n+1}(X, A, x_0) \cong [(D^{n+1}, S^n, *), (X, A, x_0)]_*\,.$$