Homology Theory

## EXERCISES

## Sheet 4, 02.10.2019

Exercise 4.1. Prove the following assertions.

- (a) The composition of two fibrations is a fibration.
- (b) The product of two fibrations is a fibration.
- (c) If  $p: E \to B$  is a fibration with B path-connected and  $E \neq \emptyset$ , then p is surjective.
- (d) Assume given a pull-back square



where p is a fibration. Prove that q is also a fibration. In the pointed case, prove that the restriction of f induces a homeomorphism  $Z \to F$ , where Z and F are the fibres of q and p, respectively.

**Exercise 4.2.** Consider the triangle  $E \subset \mathbb{R}^2$  with vertices (0,0), (1,0) und (0,1). Let B = [0,1], and let  $p : E \to B$  be the restriction of the projection  $\mathbb{R}^2 \to \mathbb{R}$ ,  $(x, y) \mapsto x$ . Prove the following assertions.

- (a) The map p is a fibration.
- (b) The map p is not a fibre bundle.

**Exercise 4.3.** Consider  $n \ge 1$ . Compute  $\pi_m(\mathbb{R}P^n, *)$  and  $\pi_\ell(\mathbb{C}P^n, *)$  for as many values of m and  $\ell$  as possible.

**Exercise 4.4.** Show that the nerve functor  $N : CAT \to \widehat{\Delta}$  is fully faithful.

**Exercise 4.5.** Let  $X \in \widehat{\Delta}$ . Recall that an *n*-simplex  $x \in X_n$  corresponds by Yoneda to a morphism  $x : \Delta_n \to X$ . An *n*-simplex *x* is called *degenerate* if it factors through some  $\Delta_m$  for m < n (and is called *non-degenerate* otherwise).

Show that for any *n*-simplex  $x : \Delta_n \to X$  there exists a unique pair  $(y, \alpha)$ , where  $\alpha : \Delta_n \to \Delta_m$  is a surjection and  $y : \Delta_m \to X$  an *m*-simplex such that  $y \circ \alpha = x$  (here we identify  $\Delta$  as a full subcategory of  $\widehat{\Delta}$ ).

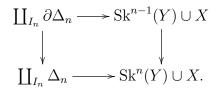
*Hint:* for the uniqueness, use that in  $\Delta$  the surjections admit a section.

**Exercise 4.6.** Let  $X \in \widehat{\Delta}$  and  $n \in \mathbb{N}$ . We define the *nth-skeleton of* S, denoted  $\operatorname{Sk}^n(X) \in \widehat{\Delta}$ , as the simplicial subset of X generated by  $X_n$  (i.e. the smallest simplicial subset of X whose set of *n*-simplices is  $X_n$ ).

- (a) Show that  $\operatorname{Sk}^n(X)_m = \{x\alpha \in X_m \mid \alpha : \Delta_m \to \Delta_n, x \in X_n\}.$
- (b) Let  $\partial \Delta_n = \bigcup_{0 \le i \le n} \text{Bild}(\delta^i : \Delta_{n-1} \to \Delta_n)$ . Let  $X \subset Y$  be a simplicial subset, and let  $I_n$  be the set of non-degenerate *n*-simplices of Y that are not in X. Prove that the obvious

http://www.math.univ-paris13.fr/~ausoni/m2-2019.html

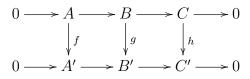
square in  $\widehat{\Delta}$  of the following form is co-carthesian:



**Exercise 4.7.** Two maps  $f_0, f_1 : X \to Y$  in  $\widehat{\Delta}$  are called *homotopic* if there is a morphism  $X \times \Delta_1 \to Y$  with  $H \circ (\operatorname{id} \times d_1) = f_0$  and  $H \circ (\operatorname{id} \times d_0) = f_1$ . We call H a *a simplicial homotopy* from  $f_0$  to  $f_1$ .

- (a) Show that if  $F_0, F_1 : C \to D \in CAT$ , a simplicial homotopy from  $NF_0$  to  $NF_1$  corresponds precisely to a natural transformation  $F_0 \Longrightarrow F_1$ .
- (b) Show that if C has a terminal object c (meaning that C(a, c) is a singleton for any  $a \in C$ ), then NC is contractible.

**Exercise 4.8.** Suppose given a commutative diagram of *R*-modules with exact rows



Prove the existence of a natural homomorphism  $\partial$ :  $\operatorname{Ker}(h) \to \operatorname{Coker}(f)$  fitting in an exact sequence

$$0 \to \operatorname{Ker}(f) \to \operatorname{Ker}(g) \to \operatorname{Ker}(h) \xrightarrow{\partial} \operatorname{Coker}(f) \to \operatorname{Coker}(g) \to \operatorname{Coker}(h) \to 0.$$

Exercise 4.9. Suppose given a commutative diagram of *R*-modules with exact rows

$$\cdots \xrightarrow{\partial_{n+1}} A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{i_{n-1}} \cdots$$

$$f_n \bigvee \qquad g_n \bigvee \qquad h_n \bigvee \qquad f_{n-1} \bigvee \qquad f_{n-1} \bigvee \qquad \cdots$$

$$\cdots \xrightarrow{\partial'_{n+1}} A'_n \xrightarrow{i'_n} B'_n \xrightarrow{p'_n} C'_n \xrightarrow{\partial'_n} A'_{n-1} \xrightarrow{i'_{n-1}} \cdots ,$$

such that  $h_n$  is an isomorphism for all n. Let  $D_n = \partial_n h_n^{-1} p'_n : B'_n \to A_{n-1}$ . Prove that the following sequence is exact:

$$\cdots \to A_n \xrightarrow{(f_n, i_n)} A'_n \oplus B_n \xrightarrow{i'_n - g_n} B'_n \xrightarrow{D_n} A_{n-1} \to \cdots$$