EXERCISES

Sheet 4, 02.10.2019

Exercise 4.1. Prove the following assertions.

(a) The composition of two fibrations is a fibration.
(b) The product of two fibrations is a fibration.
(c) If $p : E \to B$ is a fibration with B path-connected and $E \neq \emptyset$, then p is surjective.
(d) Assume given a pull-back square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ Y \arrow[r, "g"] & B \end{tikzcd}$$

where p is a fibration. Prove that q is also a fibration. In the pointed case, prove that the restriction of f induces a homeomorphism $Z \to F$, where Z and F are the fibres of q and p, respectively.

Exercise 4.2. Consider the triangle $E \subset \mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$ und $(0,1)$. Let $B = [0,1]$, and let $p : E \to B$ be the restriction of the projection $\mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x$. Prove the following assertions.

(a) The map p is a fibration.
(b) The map p is not a fibre bundle.

Exercise 4.3. Consider $n \geq 1$. Compute $\pi_m(\mathbb{R}P^n, *)$ and $\pi_\ell(\mathbb{C}P^n, *)$ for as many values of m and ℓ as possible.

Exercise 4.4. Show that the nerve functor $N : \mathrm{CAT} \to \widehat{\mathbf{\Delta}}$ is fully faithful.

Exercise 4.5. Let $X \in \widehat{\mathbf{\Delta}}$. Recall that an n-simplex $x \in X_n$ corresponds by Yoneda to a morphism $x : \Delta_n \to X$. An n-simplex x is called *degenerate* if it factors through some Δ_m for $m < n$ (and is called *non-degenerate* otherwise).

Show that for any n-simplex $x : \Delta_n \to X$ there exists a unique pair (y, α), where $\alpha : \Delta_n \to \Delta_m$ is a surjection and $y : \Delta_m \to X$ an m-simplex such that $y \circ \alpha = x$ (here we identify $\mathbf{\Delta}$ as a full subcategory of $\widehat{\mathbf{\Delta}}$).
Hint: for the uniqueness, use that in $\mathbf{\Delta}$ the surjections admit a section.

Exercise 4.6. Let $X \in \widehat{\mathbf{\Delta}}$ and $n \in \mathbb{N}$. We define the *nth-skeleton of S*, denoted $\mathrm{Sk}^n(X) \in \widehat{\mathbf{\Delta}}$, as the simplicial subset of X generated by X_n (i.e. the smallest simplicial subset of X whose set of n-simplices is X_n).

(a) Show that $\mathrm{Sk}^n(X)_m = \{x\alpha \in X_m \,|\, \alpha : \Delta_m \to \Delta_n, \quad x \in X_n\}$.
(b) Let $\partial\Delta_n = \bigcup_{0 \leq i \leq n} \mathrm{Bild}(\delta^i : \Delta_{n-1} \to \Delta_n)$. Let $X \subset Y$ be a simplicial subset, and let I_n be the set of non-degenerate n-simplices of Y that are not in X. Prove that the obvious

http://www.math.univ-paris13.fr/∼ausoni/m2-2019.html square in $\widehat{\mathbf{\Delta}}$ of the following form is co-carthesian:

$$\begin{tikzcd} \coprod_{I_n} \partial\Delta_n \arrow[r] \arrow[d] & \mathrm{Sk}^{n-1}(Y) \cup X \arrow[d] \\ \coprod_{I_n} \Delta_n \arrow[r] & \mathrm{Sk}^{n}(Y) \cup X. \end{tikzcd}$$

Exercise 4.7. Two maps $f_0, f_1 : X \to Y$ in $\widehat{\mathbf{\Delta}}$ are called *homotopic* if there is a morphism $X \times \Delta_1 \to Y$ with $H \circ (\mathrm{id} \times d_1) = f_0$ and $H \circ (\mathrm{id} \times d_0) = f_1$. We call H a *a simplicial homotopy from f_0 to f_1*.

(a) Show that if $F_0, F_1 : C \to D \in \mathrm{CAT}$, a simplicial homotopy from NF_0 to NF_1 corresponds precisely to a natural transformation $F_0 \Longrightarrow F_1$.

(b) Show that if C has a terminal object c (meaning that $C(a, c)$ is a singleton for any $a \in C$), then NC is contractible.

Exercise 4.8. Suppose given a commutative diagram of R-modules with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

Prove the existence of a natural homomorphism $\partial : \mathrm{Ker}(h) \to \mathrm{Coker}(f)$ fitting in an exact sequence

$$0 \to \mathrm{Ker}(f) \to \mathrm{Ker}(g) \to \mathrm{Ker}(h) \xrightarrow{\partial} \mathrm{Coker}(f) \to \mathrm{Coker}(g) \to \mathrm{Coker}(h) \to 0\,.$$

Exercise 4.9. Suppose given a commutative diagram of R-modules with exact rows

$$\begin{tikzcd} \cdots \arrow[r, "\partial_{n+1}"] & A_n \arrow[r, "i_n"] \arrow[d, "f_n"'] & B_n \arrow[r, "p_n"] \arrow[d, "g_n"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "h_n"'] & A_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "f_{n-1}"'] & \cdots \\ \cdots \arrow[r, "\partial'_{n+1}"] & A'_n \arrow[r, "i'_n"] & B'_n \arrow[r, "p'_n"] & C'_n \arrow[r, "\partial'_n"] & A'_{n-1} \arrow[r, "i'_{n-1}"] & \cdots, \end{tikzcd}$$

such that h_n is an isomorphism for all n. Let $D_n = \partial_n h_n^{-1} p'_n : B'_n \to A_{n-1}$. Prove that the following sequence is exact:

$$\cdots \to A_n \xrightarrow{(f_n, i_n)} A'_n \oplus B_n \xrightarrow{i'_n - g_n} B'_n \xrightarrow{D_n} A_{n-1} \to \cdots\,.$$