EXERCISES

Sheet 5, 09.10.2019

Exercise 5.1. Let $\mathbf{Ch}_R := \mathbf{Ch}(\mathbf{Mod}_R)$ be the category of chain complexes of left R-modules. Prove the following assertions:

(a) The relation $f_* \simeq g_*$ (there is a chain homotopy from f_* to g_*) on $\mathbf{Ch}_R(C_*, D_*)$ is an equivalence relation.

(b) Given $f_* \overset{H}{\simeq} g_*$ in $\mathbf{Ch}_R(C_*, D_*)$ and $f'_* \overset{H'}{\simeq} g'_*$ in $\mathbf{Ch}_R(D_*, E_*)$, we have $f'_* f_* \overset{K}{\simeq} g'_* g_*$ in $\mathbf{Ch}_R(C_*, E_*)$. Give a formula for K. Deduce the existence of a *homotopy category* $\mathbf{HoCh}_R$. A chain map $f_* : C_* \to D_*$ that is an isomorphism in $\mathbf{HoCh}_R$ is called a *chain homotopy equivalence.*

(c) A morphism $f_* : C_* \to D_*$ in $\mathbf{Ch}_R$ that induces an isomorphism in homology is called a *quasi-isomorphism*. Show that a quasi-isomorphismus is not necessary a chain homotopy equivalence.

(d) Show that if R is a field, then any quasi-isomorphism $f_* : C_* \to D_*$ is a chain homotopy equivalence.

Exercise 5.2. Let $(X, A) \in \mathbf{Top}^2$. Define

$$W(X, A) = \{X_\lambda \in \pi_0(X) \mid X_\lambda \cap A = \emptyset\} .$$

Prove that the functors $H_0(-; \mathbb{Z})$ and $\mathbb{Z}\{W(-)\} : \mathbf{Top}^2 \to \mathbf{Ab}$ are naturally isomorphic.

Exercise 5.3. Consider the following subspaces of $\mathbb{R}^2$:

$$X = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le (2\pi)^{-1}\} \subset \mathbb{R}^2,$$
$$C = \{x \in X, \mid x_1 = 0, \quad x_2 \ge -1\} \cup \{x \in X, \mid x_1 > 0, x_2 \ge \sin(1/x)\} \subset X$$
$$D = \{x \in X, \mid x_1 = 0, \quad x_2 \le 1\} \cup \{x \in X, \mid x_1 > 0, \quad x_2 \le \sin(1/x)\} \subset X.$$

Prove the following assertions:

(a) C and D are closed subspaces of X and $X = C \cup D$.

(b) The "Mayer-Vietoris" sequence

$$\cdots \to H_n(C \cap D; \mathbb{Z}) \to H_n(C; \mathbb{Z}) \oplus H_n(D; \mathbb{Z}) \to H_n(X; \mathbb{Z}) \to H_{n-1}(C \cap D; \mathbb{Z}) \to \cdots$$

is not exact.

Exercise 5.4. Compute $H_*(S^1 \times S^1; \mathbb{Z})$.

http://www.math.univ-paris13.fr/~ausoni/m2-2018.html

Exercise 5.5. Suppose $n \in \mathbb{N}$ with $n \geq 1$, and consider a map $f : S^n \to S^n$. Define the *degree of* f as the number $\deg(f) \in \mathbb{Z}$ given by the rule

$$f_* : H_n(S^n; \mathbb{Z}) \to H_n(S^n; \mathbb{Z}) , \quad i_n \mapsto \deg(f) i_n ,$$

where $i_n \in H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$ is a chosen generator. Prove the following assertions:

(a) The degree determines a well-defined map $\deg : [S^n, S^n] \to \mathbb{Z}$.
(b) If f is homotopic to a constant map, then $\deg(f) = 0$.
(c) $\deg(\mathrm{id}_{S^n}) = 1$, and deg is surjective.
(d) $\deg(fg) = \deg(f) \deg(g)$.
(e) If f is a homotopy equivalence, that $\deg(f) = \pm 1$.

Exercise 5.6. Let $n \geq 1$, and let $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a linear isomorphism. Let $\phi_f : S^n \to S^n$ be defined by $\phi_f(x) = f(x)/\|f(x)\|$. Prove that $\deg(\phi_f) = \det(f)/|\det(f)|$.

Exercise 5.7. Determine which abelian groups A, up to ismorphism, can appear in the short exact sequences of the followind type:

(a) $0 \to \mathbb{Z} \to A \to \mathbb{Z}/n \to 0$, $n \in \mathbb{N}$,
(b) $0 \to \mathbb{Z}/n \to A \to \mathbb{Z} \to 0$, $n \in \mathbb{N}$,
(c) $0 \to \mathbb{Z}/p^m \to A \to \mathbb{Z}/p^n \to 0$, where p is a prime number.

Exercise 5.8. Suppose $n \in \mathbb{N}$, $n \geq 1$. Prove that there exists a map $g : S^{2n-1} \to \mathbb{C}P^{n-1}$ such that $\mathbb{C}P^n \cong \mathbb{C}P^{n-1} \sqcup_g D^{2n}$. Compute $H_*(\mathbb{C}P^n; \mathbb{Z})$.

Exercise 5.9. Let K be the Klein bottle. Determine a map $f : S^1 \to S^1 \vee S^1$ such that $K \cong (S^1 \vee S^1) \sqcup_f D^2$, and compute $H_*(K; \mathbb{Z})$.