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On the algebraic K-theory of the complex K-theory spectrum

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Abstract Let $p \geq 5$ be a prime, let ku be the connective complex K-theory spectrum, and let $K(ku)$ be the algebraic K-theory spectrum of ku. In this paper we study the p-primary homotopy type of the spectrum $K(ku)$ by computing its mod (p, v_1) homotopy groups. We show that up to a finite summand, these groups form a finitely generated free module over the polynomial algebra $\mathbb{F}_p[b]$, where b is a class of degree $2p+2$ defined as a "higher Bott element".

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1 Introduction

The algebraic K-theory of a local or global number field F, with suitable finite coefficients, is known to satisfy a form of Bott periodicity. Bott periodicity refers here to the periodicity of topological complex K-theory, and is an example of v_1-periodicity in the sense of stable homotopy theory. For example, if p is an odd prime and if F contains a primitive p-th root of unity, then the mod (p) algebraic K-theory $K_*(F;\mathbb{Z}/p)$ of F contains a non-nilpotent Bott element β of degree 2, with

$$\beta^{p-1} = v_1.$$

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In one of its reformulations [19, 41], the Lichtenbaum-Quillen Conjecture asserts that the localization

$$K_*(F;\mathbb{Z}/p) \to K_*(F;\mathbb{Z}/p)[\beta^{-1}]$$

away from β is an isomorphism in positive degrees. In particular, $K_*(F;\mathbb{Z}/p)$ is periodic of period 2 in positive degrees. In the local case, this follows from [23, Theorem D].

The p-local stable homotopy category also features higher forms of periodicity [25], one for each integer $n \geq 0$, referred to as v_n-periodicity. It is detected for example by the nth Morava K-theory $K(n)$, having coefficients $K(0)_* = \mathbb{Q}$ and $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$ with $|v_n| = 2p^n - 2$ if $n \geq 1$. The study of v_2-periodicity is at the focus of current research in algebraic topology, as illustrated for example by the efforts to define the elliptic cohomology theory known as topological modular forms [24].

Waldhausen [44] extended the definition of algebraic K-theory to include specific "rings up to homotopy" called structured ring spectra, like E_∞ ring spectra [30], S-algebras [20], or symmetric ring-spectra [26]. The chromatic red-shift conjecture [4] of John Rognes predicts that the algebraic K-theory of a suitable v_n-periodic structured ring-spectrum is essentially v_{n+1}-periodic, as illustrated above in the case of number fields (which are v_0-periodic). For an example with the next level of periodicity, we consider the algebraic K-theory of topological K-theory.

Let $p \geq 5$ be a prime, and let ku_p denote the p-completed connective complex K-theory spectrum with coefficients $ku_{p*} = \mathbb{Z}_p[u]$, $|u| = 2$, where $\mathbb{Z}_p$ is the ring of p-adic integers. Let ℓ_p be the Adams summand of ku_p with coefficients $\ell_{p*} = \mathbb{Z}_p[v_1]$ and $v_1 = u^{p-1}$. In joint work with John Rognes [2], we have computed the mod (p, v_1) algebraic K-theory of the S-algebra ℓ_p, denoted $V(1)_*K(\ell_p)$, and we have shown that it is essentially v_2-periodic. This computation provides a first example of red-shift for non-ordinary rings.

In this paper, following the discussion in [1, Sect. 10], we interpret ku_p as a tamely ramified extension of ℓ_p of degree $p-1$, and we compute $V(1)_*K(ku_p)$. As expected, the result is again essentially periodic. However, $V(1)_*K(ku_p)$ has a shorter period: its periodicity is given by multiplication with a higher Bott element $b \in V(1)_*K(ku_p)$, of degree $2p+2$. We defer a definition of b to Sect. 3 below, and summarize our main result in the following statement.

Theorem 1.1 *Let $p \geq 5$ be a prime. The higher Bott element $b \in V(1)_{2p+2}K(ku_p)$ is non-nilpotent and satisfies the relation*

$$b^{p-1} = -v_2.$$

Let $P(b)$ denote the polynomial $\mathbb{F}_p$-sub-algebra of $V(1)_*K(ku_p)$ generated by b. Then there is a short exact sequence of graded $P(b)$-modules

$$0 \to \Sigma^{2p-3}\mathbb{F}_p \to V(1)_*K(ku_p) \to F \to 0,$$

where $\Sigma^{2p-3}\mathbb{F}_p$ is the sub-module of b-torsion elements and F is a free $P(b)$-module on $8+4(p-1)$ generators.

A detailed description of the free $P(b)$-module F is given in Theorem 8.1. The proof is based on evaluating the cyclotomic trace map [11]

$$\mathrm{trc} : K(ku_p) \to TC(ku_p)$$

to topological cyclic homology. We emphasize that the higher Bott element b is not the reduction of a class in the mod (p) or integral homotopy of $K(ku_p)$.

The cyclic subgroup $\Delta \subset \mathbb{Z}_p^\times$ of order $p-1$ acts on ku_p by p-adic Adams operations. The Adams summand is defined as the homotopy fixed-point spectrum $\ell_p = ku_p^{h\Delta}$, and Δ qualifies as the Galois group of the tamely ramified extension $\ell_p \to ku_p$ of commutative S-algebras given by the inclusion of homotopy fixed-points. We proved in [1, Theorem 10.2] that the induced map $K(\ell_p) \to K(ku_p)$ factors through a weak equivalence

$$K(\ell_p) \xrightarrow{\simeq} K(ku_p)^{h\Delta}$$

after p-completion. The mod (p, v_1) homotopy groups of $K(\ell_p)$ and $K(ku_p)$ are related as follows.

Proposition 1.2 *Let $i_* : V(1)_*K(\ell_p) \to V(1)_*K(ku_p)$ be the homomorphism induced by the extension of S-algebras $\ell_p \to ku_p$.*

(a) *The homomorphism i_* factors through an isomorphism*

$$V(1)_*K(\ell_p) \cong \big(V(1)_*K(ku_p)\big)^{\Delta} \subset V(1)_*K(ku_p)$$

onto the classes fixed by the Galois group. The higher Bott element b is not fixed under the action of Δ, but $b^{p-1} = -v_2$ is, accounting for the v_2-periodicity of $V(1)_*K(\ell_p)$.

(b) *The homomorphism*

$$\mu : P(b) \otimes_{P(v_2)} V(1)_*K(\ell_p) \to V(1)_*K(ku_p)$$

induced by i_* and the $P(b)$-action has finite kernel and cokernel, and is an isomorphism in degrees larger than $2p^2-4$. By localizing away from b, we obtain an isomorphism of $P(b, b^{-1})$-modules

$$P(b, b^{-1}) \otimes_{P(v_2)} V(1)_*K(\ell_p) \xrightarrow{\cong} V(1)_*K(ku_p)[b^{-1}].$$

In particular, the $P(b)$-module $V(1)_*K(ku_p)$ is almost the module obtained from the $P(v_2)$-module $V(1)_*K(\ell_p)$ by the extension $P(v_2) \subset P(b)$ of scalars. The kernel of μ consists of b-multiples of the v_2-torsion elements, and we have a non-trivial cokernel because some of the $P(v_2)$-module generators of $V(1)_*K(\ell_p)$ are multiples of b in $V(1)_*K(ku_p)$, see Corollary 8.2.

Notice that for the cyclotomic extension $\mathbb{Z}_p \to \mathbb{Z}_p[\zeta_p]$ of complete discrete valuation rings with Galois group Δ (where ζ_p is a primitive pth root of unity), we have corresponding results in mod (p) algebraic K-theory. In effect, the natural homomorphism $K_*(\mathbb{Z}_p;\mathbb{Z}/p) \to K_*(\mathbb{Z}_p[\zeta_p];\mathbb{Z}/p)$ factors through an isomorphism onto the Δ-fixed classes. The Bott class $\beta \in K_2(\mathbb{Z}_p[\zeta_p];\mathbb{Z}/p)$ is not fixed under Δ, but $\beta^{p-1} = v_1$ is. This accounts for the fact that $K_*(\mathbb{Z}_p[\zeta_p];\mathbb{Z}/p)$ has a shorter period than $K_*(\mathbb{Z}_p;\mathbb{Z}/p)$. Moreover, the $P(\beta)$-module $K_*(\mathbb{Z}_p[\zeta_p];\mathbb{Z}/p)$ is essentially obtained from the $P(v_1)$-module $K_*(\mathbb{Z}_p;\mathbb{Z}/p)$ by the extension $P(v_1) \subset P(\beta)$ of scalars. These facts are extracted from computations by Hesselholt and Madsen [23, Theorem D]. We therefore interpret Proposition 1.2 as follows: up to a chromatic shift of one in the sense of stable homotopy theory, the algebraic K-theory spectra of the tamely ramified extensions

$$\begin{array}{ccc} \mathbb{Z}_p[\zeta_p] & & ku_p \\ \Delta \uparrow & \text{and} & \uparrow \Delta \\ \mathbb{Z}_p & & \ell_p \end{array}$$

have a comparable formal structure.

This example of red-shift provides evidence that structural results for the algebraic K-theory of ordinary rings might well be generalized to provide more conceptual descriptions of the algebraic K-theory of S-algebras. See Remarks 3.5 and 8.4 for a discussion of the results we have in mind here.

We now turn to the algebraic K-theory $K(ku)$ of the (non p-completed) connective complex K-theory spectrum ku, with coefficients $ku_* = \mathbb{Z}[u]$, $|u| = 2$. The p-completion $ku \to ku_p$ induces a map

$$\kappa : K(ku) \to K(ku_p),$$

and the higher Bott element $b \in V(1)_{2p+2}K(ku_p)$ is in fact defined as the image of a class with same name in $V(1)_{2p+2}K(ku)$. The difference between $K(ku)$ and $K(ku_p)$ can be measured by means of the homotopy Cartesian square after p-completion

$$\begin{tikzcd} K(ku) \arrow[r, "\pi"] \arrow[d, "\kappa"'] & K(\mathbb{Z}) \arrow[d, "\kappa"] \\ K(ku_p) \arrow[r, "\pi"] & K(\mathbb{Z}_p) \end{tikzcd}$$

of Dundas [18, p. 224]. Here π denotes the map induced in K-theory by the zeroth Postnikov sections $ku \to H\mathbb{Z}$ and $ku_p \to H\mathbb{Z}_p$, where HR is the Eilenberg-Mac Lane spectrum of the ring R. The homotopy type of the p-completion of $K(\mathbb{Z}_p)$ has been computed by Bökstedt, Hesselholt and Madsen [10, 22]. The Lichtenbaum-Quillen Conjecture for $K(\mathbb{Z})$ (see for example [34, Chap. 6]) implies that the homotopy fiber of $K(\mathbb{Z}) \to K(\mathbb{Z}_p)$ has finite $V(1)$-homotopy groups, which are concentrated in degrees smaller than $2p-1$. This implies the result below. In fact there seems to be some consensus that work of Vladimir Voevodsky and Markus Rost should imply the Lichtenbaum-Quillen Conjecture, but to our knowledge this has not appeared in written form. We therefore keep it as an assumption in the following results.

Proposition 1.3 *Let $p \geq 5$ be a prime, and assume that the Lichtenbaum-Quillen Conjecture for $K(\mathbb{Z})$ holds at p. Then the homomorphism of $P(b)$-modules*

$$\kappa_* : V(1)_* K(ku) \to V(1)_* K(ku_p)$$

is an isomorphism in degrees larger than $2p-1$. Localizing the $V(1)$-homotopy groups away from b, we obtain an isomorphism

$$V(1)_* K(ku)[b^{-1}] \cong V(1)_* K(ku_p)[b^{-1}]$$

of $P(b, b^{-1})$-algebras.

This result is of interest beyond algebraic K-theory. Baas, Dundas and Rognes have proposed a geometric definition of a cohomology theory derived from a suitable notion of bundles of complex two-vector spaces [6]. These are a two-categorical analogue of the ordinary complex vector bundles which enter in the geometric definition of topological K-theory. They conjectured in [6, 5.1] that the spectrum representing this new theory is weakly homotopy equivalent to $K(ku)$, and this was proved by these authors and Birgit Richter in [7]. The next statement follows from Theorem 1.1 and Proposition 1.3.

Proposition 1.4 *If the Lichtenbaum-Quillen Conjecture for $K(\mathbb{Z})$ holds, then at any prime $p \geq 5$ the spectrum $K(ku)$ is of telescopic complexity two in the sense of* [6, 6.1].

This result was anticipated in [6, Sect. 6], and ensures that the cohomology theory derived from two-vector bundles is, from the view-point of stable homotopy theory, a legitimate candidate for elliptic cohomology.

The computations presented in this paper fail at the primes 2 and 3, because of the non-existence of the ring-spectrum $V(1)$. Theoretically, computations in mod (p) homotopy or in integral homotopy could also be carried out, but the algebra seems quite intractable. Another approach [16, 28] is via homology computations. There are ongoing projects in this direction by Robert Bruner, Sverre Lunøe-Nielsen and John Rognes.

Up to degree three, the integral homotopy groups of $K(ku)$ can be computed essentially by using the map $\pi : K(ku) \to K(\mathbb{Z})$ introduced above. The map $\pi_* : K_*(ku) \to K_*(\mathbb{Z})$ is 3-connected, so that

$$K_0(ku) \cong \mathbb{Z}, \qquad K_1(ku) \cong \mathbb{Z}/2 \quad \text{and} \quad K_2(ku) \cong \mathbb{Z}/2.$$

Here $K_1(ku)$ and $K_2(ku)$ are generated by the image of $\eta \in \pi_1 S$ and $\eta^2 \in \pi_2 S$, respectively, under the unit $S \to K(ku)$. Let $w : BBU_\otimes \to \Omega^\infty K(ku)$ be the map induced by the inclusion of units, see (3.3). There is a non-split extension

$$0 \to \pi_3(BBU_\otimes) \xrightarrow{w_*} K_3(ku) \xrightarrow{\pi_*} K_3(\mathbb{Z}) \to 0$$

with $\pi_3(BBU_\otimes) \cong \mathbb{Z}\{\mu\}$, $K_3(ku) \cong \mathbb{Z}\{\varsigma\} \oplus \mathbb{Z}/24\{\nu\}$ and $K_3(\mathbb{Z}) \cong \mathbb{Z}/48\{\lambda\}$, where ν is the image of the Hopf class ν, which generates $\pi_3 S \cong \mathbb{Z}/24$. We have $w_*(\mu) = 2\varsigma - \nu$ and $\pi_*(\varsigma) = \lambda$. See [5] for details. This indicates that the integral homotopy groups $K_*(ku)$ contain intriguing non-trivial extensions from subgroups in $\pi_* S$, $\pi_* BBU_\otimes$ and $K_*(\mathbb{Z})$.

The rational algebraic K-groups of ku are well understood. In joint work with Rognes [3], we have proved that after rationalization, the sequence

$$BBU_\otimes \xrightarrow{w} \Omega^\infty K(ku) \xrightarrow{\pi} \Omega^\infty K(\mathbb{Z})$$

is a split homotopy fibre-sequence. A rational splitting of w is provided by a rational determinant map $\Omega^\infty K(ku) \to (BBU_\otimes)_\mathbb{Q}$. In particular, by Borel's computation [14] of $K_*(\mathbb{Z}) \otimes \mathbb{Q}$, there is a rational equivalence

$$\Omega^\infty K(ku) \simeq_\mathbb{Q} SU \times (SU/SO) \times \mathbb{Z}.$$

All but finitely many of the non-torsion classes in the integral homotopy groups $\pi_* K(ku)$ detected by this equivalence reduce mod (p) to multiples of v_1, and hence reduce to zero in $V(1)_* K(ku)$.

We briefly discuss the contents of this paper. In Sect. 2, we study the $V(1)$-homotopy of the Eilenberg-Mac Lane space $K(\mathbb{Z},3)$, which is a subspace of the space of units of ku. In Sect. 3, we define low-dimensional classes in $V(1)_*K(ku)$ corresponding to units of ku, and in particular we introduce the higher Bott element. We prove in Sect. 4 that these classes are non-zero by means of the Bökstedt trace map

$$\mathrm{tr}: K(ku) \to THH(ku)$$

to topological Hochschild homology. In Sect. 5, we compute $V(1)_n K(ku_p)$ for $n \leq 2p-2$. This complements the computations in higher degrees provided by the cyclotomic trace

$$\mathrm{trc}: K(ku_p) \to TC(ku_p)$$

to topological cyclic homology. In Sect. 6 we compute the various homotopy fixed points of $THH(ku_p)$ under the action of the cyclic groups C_{p^n} and the circle, which are the ingredients for the computation of $V(1)_*TC(ku_p)$ in Sect. 7. In Sect. 8 we prove Theorem 1.1 on the structure of $V(1)_*K(ku_p)$ stated above. We also give a computation of $V(1)_*K(KU_p)$ for KU_p the p-completed periodic K-theory spectrum, up to some indeterminacy.

Notations and conventions Throughout the paper, unless stated otherwise, p will be a fixed prime with $p \geq 5$, and $\mathbb{Z}_p$ will denote the p-adic integers. For an $\mathbb{F}_p$-vector space V, let $E(V)$, $P(V)$ and $\Gamma(V)$ be the exterior algebra, polynomial algebra and divided power algebra on V, respectively. If V has a basis $\{x_1, \dots, x_n\}$, we write $V = \mathbb{F}_p\{x_1, \dots, x_n\}$ and $E(x_1, \dots, x_n)$, $P(x_1, \dots, x_n)$ and $\Gamma(x_1, \dots, x_n)$ for these algebras. By definition, $\Gamma(x)$ is the $\mathbb{F}_p$-vector space $\mathbb{F}_p\{\gamma_k x \mid k \geq 0\}$ with product given by $\gamma_i x \cdot \gamma_j x = \binom{i+j}{i}\gamma_{i+j}x$, where $\gamma_0 x = 1$ and $\gamma_1 x = x$. Let $P_h(x) = P(x)/(x^h)$ be the truncated polynomial algebra of height h. For an algebra A, we denote by $A\{x_1, \dots, x_n\}$ the free A-module generated by $x_1, \dots, x_n$.

If Y is a space and E_* is a homology theory, such as mod (p) homology, $V(1)$-homotopy or Morava K-theory $K(2)_*$, we denote by $E_*(Y)$ the unreduced E_*-homology of Y, which we identify with the E_*-homology of the suspension spectrum $\Sigma^\infty(Y_+)$, where Y_+ denotes Y with a disjoint base-point added. We usually write $\Sigma^\infty_+ Y$ instead of $\Sigma^\infty(Y_+)$.

The reduced E_*-homology of a pointed space X is denoted $\widetilde{E}_*(X)$. We denote $\pi_* X$ the (unstable) homotopy groups of X, and $\pi_*\Sigma^\infty X$ its stable homotopy groups.

If $f: A \to B$ is a map of S-algebras, we also denote by f its image under various functors like THH, TC or K.

In our computations with spectral sequences, we often determine a differential d only up to multiplication by a unit. We use the notation $d(x) \doteq y$

to indicate that the equation $d(x) = \alpha y$ holds for some unit $\alpha \in \mathbb{F}_p$. Classes surviving to the E^r-term of a spectral sequence, for $r \geq 3$, are often given as a product of classes in the E^2-term. To improve the readability, we denote the product of two classes x, y in E^r by $x \cdot y$.

2 On the $V(1)$-homotopy of $K(\mathbb{Z}, 3)$

If G is a topological monoid, let us denote by BG its classifying space, obtained by realization of the bar construction, see for example [36, Chap. 1]. If G is an Abelian topological group, then so is BG. The space BG is equipped with the bar filtration

$$\{*\} = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_{n-1} \subset B_n \subset \cdots BG, \tag{2.1}$$

with filtration quotients $B_n/B_{n-1} \cong \Sigma^n(G^{\wedge n})$. In particular, we have a map

$$s : \Sigma G = B_1 \subset BG, \tag{2.2}$$

which in any homology theory E_* induces a map

$$\sigma : E_*G \to E_{*+1}BG$$

called the *suspension*. If E_* is a multiplicative homology theory satisfying the Künneth isomorphism, we have the bar spectral sequence [36, Chap. 2]

$$E^1_{s,*}(G) = \widetilde{E}_*(G)^{\otimes_{E_*} s},$$
$$E^2_{s,t}(G) = \mathrm{Tor}^{E_*(G)}_{s,t}(E_*, E_*) \Rightarrow E_{s+t}(BG)$$

associated to the bar filtration (2.1).

Let $K(\mathbb{Z}, 0)$ be equal to $\mathbb{Z}$ as a discrete topological group, and for $m \geq 1$, we define recursively the Eilenberg-Mac Lane space $K(\mathbb{Z}, m)$ as the Abelian topological group $BK(\mathbb{Z}, m-1)$. We recall Cartan's computation of the algebra $H_*(K(\mathbb{Z}, m); \mathbb{F}_p)$ for p an odd prime and $m = 2, 3$. The generators are constructed explicitly from the unit $1 \in H_*(K(\mathbb{Z}, 0); \mathbb{F}_p)$ by means of the suspension σ and two further operators

$$\varphi : H_{2q}(K(\mathbb{Z}, m); \mathbb{F}_p) \to H_{2pq+2}(K(\mathbb{Z}, m+1); \mathbb{F}_p) \quad \text{and}$$
$$\gamma_p : H_{2q}(K(\mathbb{Z}, m); \mathbb{F}_p) \to H_{2pq}(K(\mathbb{Z}, m); \mathbb{F}_p),$$

called the *transpotence* [17, p. 6-06] and the *p-th divided power* [17, p. 7-07], respectively. The transpotence is an additive homomorphism since p is odd. For $x \in H_{2q}(K(\mathbb{Z}, m); \mathbb{F}_p)$, the class $\varphi(x)$ is represented, for example, by

$$x^{p-1} \otimes x \in E^1_{2,2pq}(K(\mathbb{Z}, m))$$

in the bar spectral sequence. The algebra $H_*(K(\mathbb{Z}, m); \mathbb{F}_p)$ has the structure of an algebra with divided powers, which are uniquely determined by γ_p.

Theorem 2.1 (Cartan) *Let p be an odd prime. There are isomorphisms of $\mathbb{F}_p$-algebras with divided powers*

$$\Gamma(y) \xrightarrow{\cong} H_*(K(\mathbb{Z}, 2); \mathbb{F}_p)$$

given by $y \mapsto \sigma\sigma(1)$, with $|y| = 2$, and

$$\bigotimes_{k\geq 0} E(e_k) \otimes \Gamma(f_k) \xrightarrow{\cong} H_*(K(\mathbb{Z}, 3); \mathbb{F}_p),$$

given by $e_k \mapsto \sigma\gamma_p^k\sigma\sigma(1)$ and $f_k \mapsto \varphi\gamma_p^k\sigma\sigma(1)$, with degrees $|e_k| = 2p^k + 1$ and $|f_k| = 2p^{k+1} + 2$. For $k \geq 0$, the generators f_k and e_{k+1} are related by a primary mod (p) homology Bockstein

$$\beta(f_k) = e_{k+1}.$$

Proof The computation of $H_*(K(\mathbb{Z}, m); \mathbb{F}_p)$ as an algebra is given in [17, Théorème fondamental, p. 9-03]. The Bockstein relation $\beta(f_k) = e_{k+1}$ is established in [17, p. 8-04]. □

Ravenel and Wilson [36] make use of the bar spectral sequence to compute the Morava K-theory $K(n)_*K(\pi, m)$ as an algebra when $\pi = \mathbb{Z}$ or $\mathbb{Z}/p^j$. All generators can be defined explicitly, starting with the unit $1 \in K(n)_*K(\pi, 0)$ and using the suspension, divided powers, transpotence and the Hopf-ring structure on $K(n)_*K(\pi, *)$. We refer to [36, 5.6 and 12.1] for the following result, and for the definition of the generators $\beta_{(k)}$ and $b_{(2k,1)}$.

Theorem 2.2 (Ravenel-Wilson) *Let $p \geq 3$ be a prime and let $K(2)$ be the Morava K-theory spectrum with coefficients $K(2)_* = \mathbb{F}_p[v_2, v_2^{-1}]$. There are isomorphisms of $K(2)_*$-algebras*

$$K(2)_*K(\mathbb{Z}, 2) \cong K(2)_*[\beta_{(k)} \mid k \geq 0]/(\beta_{(0)}^p, \beta_{(k+1)}^p - v_2^{p^k}\beta_{(k)} \mid k \geq 0)$$

where $|\beta_{(k)}| = 2p^k$, and

$$K(2)_*K(\mathbb{Z}, 3) \cong K(2)_*[b_{(2k,1)} \mid k \geq 0]/(b_{(2k,1)}^p + v_2^{p^k} b_{(2k,1)} \mid k \geq 0)$$

where $|b_{(2k,1)}| = 2p^k(p + 1)$. The class $\beta_{(0)} \in K(2)_2K(\mathbb{Z}, 2)$ is equal to $\sigma\sigma(1)$, and the class $b_{(0,1)} \in K(2)_{2p+2}K(\mathbb{Z}, 3)$ is the transpotence of $\beta_{(0)}$.

We now turn to $V(1)$-homotopy. For an integer $n \geq 0$, we denote by $V(n)$ the Smith-Toda complex [42], with mod (p) homology given by

$$H_*(V(n); \mathbb{F}_p) \cong E(\tau_0, \dots, \tau_n)$$

as a left sub-comodule of the dual Steenrod algebra. In particular, $V(0) = S/p$ is the mod (p) Moore spectrum, and the spectra $V(0)$ and $V(1)$ fit in cofibre sequences

$$S \xrightarrow{p} S \xrightarrow{i_0} V(0) \xrightarrow{j_0} \Sigma S$$

and

$$\Sigma^{2p-2} V(0) \xrightarrow{v_1} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{2p-1} V(0),$$

where v_1 is a periodic map. For $n = 0, 1$ and $p \geq 5$, the spectrum $V(n)$ is a commutative ring spectrum [35], and its ring of coefficients $V(n)_*$ is an $\mathbb{F}_p$-algebra which contains a non-nilpotent class v_{n+1}, of degree $2p^{n+1} - 2$. We call "$V(n)$-homotopy" the homology theory associated to the spectrum $V(n)$. In other words, the $V(n)$-homotopy groups of a spectrum X are defined by

$$V(n)_* X = \pi_*(V(n) \wedge X).$$

Notice that $V(0)_*X$ is denoted $\pi_*(X; \mathbb{Z}/p)$ by some authors, and called the mod (p) homotopy groups of X. By analogy, we sometimes call $V(1)_*X$ the mod (p, v_1) homotopy groups of X. If Y is a space, then $V(n)_*Y$ is defined as $V(n)_*\Sigma^\infty_+ Y$.

The primary mod (p) homotopy Bockstein $\beta_{0,1} : V(0)_*X \to V(0)_{*-1}X$ is the homomorphism induced by $(\Sigma i_0) j_0$, and the primary mod (v_1) homotopy Bockstein $\beta_{1,1} : V(1)_*X \to V(1)_{*-2p+1}X$ is the homomorphism induced by $(\Sigma^{2p-1} i_1) j_1$. The homomorphisms $i_{0*} : \pi_*(X) \to V(0)_*X$ and $i_{1*} : V(0)_*X \to V(1)_*X$ are called the mod (p) reduction and the mod (v_1) reduction, respectively.

Let $H\mathbb{F}_p$ be the Eilenberg-Mac Lane spectrum of $\mathbb{F}_p$. The unit map $S \to H\mathbb{F}_p$ factors through a map of ring spectra $h : V(1) \to H\mathbb{F}_p$, which induces an injective homomorphism in mod (p) homology. Identifying the homology of $V(1)$ with its image in the dual Steenrod algebra A_*, we obtain the isomorphism

$$H_*(V(1); \mathbb{F}_p) \cong E(\tau_0, \tau_1)$$

of left A_*-comodule algebras mentioned above. Toda [42, Theorem 5.2] computed $V(1)_*$ in a range of degrees for which the Adams spectral sequence collapses. Up to some renaming of the classes, we deduce from his theorem that for $p \geq 5$ there is an isomorphism of $P(v_2) \otimes P(\beta_1)$-modules

$$P(v_2) \otimes P(\beta_1) \otimes \mathbb{F}_p\{1, \alpha_1, \beta_1', (\alpha_1\beta_1)^\sharp\} \to V(1)_* \tag{2.3}$$

in degrees $* < 4p^2 - 2p - 4$. The classes α_1 and β_1 are the mod (p, v_1) reduction of the classes with same name in $\pi_*(S)$, of degrees $2p-3$ and $2p^2-2p-2$, respectively. The class β_1' is the mod (v_1) reduction of the class with same name in $V(0)_*$ that supports a primary mod (p) homotopy Bockstein $\beta_{0,1}(\beta_1') = \beta_1$, and is of degree $2p^2-2p-1$. The classes v_2 and $(\alpha_1\beta_1)^\sharp$, of degree $2p^2-2$ and $2p^2+2p-6$ respectively, support a primary mod (v_1) homotopy Bockstein, given by $\beta_{1,1}(v_2) = \beta_1'$ and $\beta_{1,1}((\alpha_1\beta_1)^\sharp) = \alpha_1\beta_1$. The class v_2 is non-nilpotent. The lowest-degree class in $V(1)_*$ that is not in the image of (2.3) is the mod (p, v_1) reduction of the class β_2 in $\pi_*(S)$, of degree $4p^2-2p-4$.

If X is a connective spectrum of finite type, the Atiyah-Hirzebruch spectral sequence

$$E^2_{s,t} = H_s(X; \mathbb{F}_p) \otimes V(1)_t \Rightarrow V(1)_{s+t}X \tag{2.4}$$

converges strongly, and we can use it to compute $V(1)_*X$ in low degrees. The first non-trivial Postnikov invariant of $V(1)$ is Steenrod's reduced power operation P^1, corresponding to the first possibly non-trivial differential of the spectral sequence on the zeroth line, see Remark 2.4. This operation detects the class α_1, which belongs to the kernel of the Hurewicz homomorphism $V(1)_* \to H_*(V(1); \mathbb{F}_p)$. In some more details, we have a commutative diagram

$$\begin{tikzcd} & V(1) \arrow[d, "\rho"] \arrow[dr, "h"] & & \\ \Sigma^{2p-3}H\mathbb{F}_p \arrow[r, "g"] & V(1)[2p-3] \arrow[r, "h"] & H\mathbb{F}_p \arrow[r, "P^1"] & \Sigma^{2p-2}H\mathbb{F}_p, \end{tikzcd} \tag{2.5}$$

where ρ is the $(2p-3)$th-Postnikov section, and the horizontal sequence is a cofibre sequence. Notice that by (2.3) the map ρ is $(2p^2-2p-2)$-connected, so that under our assumptions on X we have a well defined homomorphism

$$\alpha = (\rho_*)^{-1}g_* : H_{n-2p+3}(X; \mathbb{F}_p) \to V(1)_nX$$

for $n \leq 2p^2 - 2p - 3$.

Lemma 2.3 *Let X be a connective spectrum of finite type, and let $p \geq 3$ be a prime. For $n \leq 2p^2-2p-3$, the group $V(1)_nX$ fits in an exact sequence*

$$\begin{aligned} H_{n+1}(X; \mathbb{F}_p) &\xrightarrow{(P^1)^*} H_{n-2p+3}(X; \mathbb{F}_p) \xrightarrow{\alpha} V(1)_nX \\ &\xrightarrow{h_*} H_n(X; \mathbb{F}_p) \xrightarrow{(P^1)^*} H_{n-2p+2}(X; \mathbb{F}_p). \end{aligned}$$

Here $(P^1)^*$ *denotes the homology operation dual to* P^1*. If* X *is a ring spectrum then* α *sends the unit* $1 \in H_0(X;\mathbb{F}_p)$ *to* α_1*. Moreover, for any* X *and any* $n \geq 0$*, we have a commutative diagram*

$$\begin{array}{ccc} V(1)_n X & \xrightarrow{h_*} & H_n(X;\mathbb{F}_p) \\ {\scriptstyle \beta_{1,1}}\downarrow & & \downarrow{\scriptstyle Q_1^*} \\ V(1)_{n-2p+1} X & \xrightarrow{h_*} & H_{n-2p+1}(X;\mathbb{F}_p) \end{array}$$

relating the primary mod (v_1) *homotopy Bockstein* $\beta_{1,1}$ *to the homology operation* Q_1^* *dual to Milnor's primitive* $Q_1 = P^1\delta - \delta P^1 \in A$.

Proof This exact sequence is the sequence associated to the cofibre sequence in (2.5), where we have replaced $V(1)[2p-3]_n X$ by $V(1)_n X$ via ρ_*, which is an isomorphism for these values of n, by strong convergence of the Atiyah-Hirzebruch spectral sequence. The assertion on α_1 is true if $X = S$, and follows by naturality for X an arbitrary ring spectrum.

The self-map $f = (\Sigma^{2p-1} i_1) j_1$ of $V(1)$, which induces $\beta_{1,1}$, is given in mod (p) homology by the homomorphism $f_* : E(\tau_0, \tau_1) \to E(\tau_0, \tau_1)$ of degree $1-2p$ with $f_*(1) = f_*(\tau_0) = 0$, $f_*(\tau_1) = 1$ and $f_*(\tau_0\tau_1) = \tau_0$. We have a commutative diagram

$$\begin{array}{ccc} V(1)_* X & \xrightarrow{\beta_{1,1}} & V(1)_* X \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ E(\tau_0, \tau_1) \otimes H_*(X;\mathbb{F}_p) & \xrightarrow{f_* \otimes 1} & E(\tau_0, \tau_1) \otimes H_*(X;\mathbb{F}_p) \\ {\scriptstyle e_*}\downarrow & & \downarrow{\scriptstyle \mu} \\ A_* \otimes H_*(X;\mathbb{F}_p) & \xrightarrow{{\tau_1}^* \otimes 1} & H_*(X;\mathbb{F}_p). \end{array}$$

The horizontal arrows are of degree $1-2p$, and ${\tau_1}^* : A_* \to \mathbb{F}_p$ is the dual of τ_1 with respect to the standard basis $\{\tau(E)\xi(R)\}$ of A_* given in [33, Chap. 6]. The homomorphism g_* is induced in homotopy by the smash product of the unit $S \to H\mathbb{F}_p$ with the identity of $V(1) \wedge X$, μ is induced by the right homotopy action $H\mathbb{F}_p \wedge V(1) \to H\mathbb{F}_p$, and e_* is induced by $1 \wedge h \wedge 1 : H\mathbb{F}_p \wedge V(1) \wedge X \to H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge X$. We have $\mu g_* = h_*$ and $e_* g_* = \nu_* h_*$, where ν_* is the left A_*-coaction on $H_*(X;\mathbb{F}_p)$. This completes the proof since $({\tau_1}^* \otimes 1)\nu_* = Q_1^*$ by definition of Q_1, see [33, p. 163]. □

Remark 2.4 For X connective, the Atiyah-Hirzebruch spectral sequence (2.4) has only two non-trivial lines in internal degrees $t \leq 2p^2 - 2p - 3$, corresponding to 1 and α_1 in $V(1)_*$, see (2.3). The argument above shows that there is a differential

$$d^{2p-2}(z) = (P^1)^*(z)\alpha_1$$

for $z \in E^2_{*,0}$. In total degrees less than $2p^2 - 2p - 3$ this is the only possibly non-trivial differential.

Lemma 2.5 *The map*

$$\mathbb{F}_p\{\alpha_1\} \oplus P_p(x) \to V(1)_* K(\mathbb{Z}, 2)$$

given by $x \mapsto \sigma\sigma(1)$ with $|x| = 2$ is an isomorphism in degrees less than $4p - 3$.

Proof This follows from Theorem 2.1, Lemma 2.3 and the relation

$$(P^1)^*\big(\gamma_{k+p-1}(y)\big) = k\gamma_k(y) \tag{2.6}$$

in $H_*(K(\mathbb{Z}, 2); \mathbb{F}_p) \cong \Gamma(y)$. $\square$

Consider the cofibration

$$B_1 = \Sigma K(\mathbb{Z}, 2) \xrightarrow{i} B_2 \xrightarrow{j} \Sigma^2(K(\mathbb{Z}, 2)^{\wedge 2}) \to \Sigma^2 K(\mathbb{Z}, 2)$$

extracted from the bar filtration (2.1) of $K(\mathbb{Z}, 3)$. It induces an exact sequence

$$V(1)_*\Sigma K(\mathbb{Z}, 2) \xrightarrow{i_*} V(1)_* B_2 \xrightarrow{j_*} \widetilde{V(1)}_*\Sigma^2(K(\mathbb{Z}, 2)^{\wedge 2}) \xrightarrow{\Sigma^2\mu_*} V(1)_*\Sigma^2 K(\mathbb{Z}, 2),$$

where μ_* is induced by the product on $K(\mathbb{Z}, 2)$. We know that $V(1)_{2p+1}K(\mathbb{Z}, 2) = 0$, by Lemma 2.5, which implies that the homomorphism

$$V(1)_{2p+2}B_2 \xrightarrow{j_*} \widetilde{V(1)}_{2p}K(\mathbb{Z}, 2)^{\wedge 2}$$

is injective. We know as well that the composition

$$\widetilde{V(1)}_*K(\mathbb{Z}, 2) \otimes \widetilde{V(1)}_*K(\mathbb{Z}, 2) \xrightarrow{k} \widetilde{V(1)}_*K(\mathbb{Z}, 2)^{\wedge 2} \xrightarrow{\mu_*} V(1)_*K(\mathbb{Z}, 2)$$

sends the class $x^{p-1} \otimes x$ to zero. In particular, the class $k(x^{p-1} \otimes x)$ is in the image of j_*. Let $\tilde{b}' \in V(1)_{2p+2}B_2$ be the unique class which satisfies the equation

$$j_*(\tilde{b}') = \Sigma^2 k(x^{p-1} \otimes x).$$

Definition 2.6 We define the fundamental class $e'_0 \in V(1)_3K(\mathbb{Z},3)$ as the image of the unit $1 \in V(1)_0K(\mathbb{Z},0)$ under the iterated suspension σ^3. We define

$$b' \in V(1)_{2p+2}K(\mathbb{Z},3)$$

as $b' = l_{2*}(\tilde{b}')$, where $l_2 : B_2 \to K(\mathbb{Z},3)$ is the inclusion of the second subspace in the bar filtration.

Notice that the definition of b' in $V(1)$-homotopy, using $x^{p-1} \otimes x$ as above, lifts the definition of the transpotence in the homology of the bar construction. We use this fact in the proof of the following proposition.

Proposition 2.7 *The class $b' \in V(1)_*K(\mathbb{Z},3)$ is non-nilpotent, and satisfies the relation*

$$b'^p = -v_2 b'.$$

There is a primary mod (v_1) homotopy Bockstein

$$\beta_{1,1}(b') = e'_0.$$

Proof First, we notice that the $\mathbb{F}_p$-vector space $V(1)_{2p^2+2p}K(\mathbb{Z},3)$, which contains b'^p, is of rank at most one. Indeed, consider the Atiyah-Hirzebruch spectral sequence

$$E^2_{s,t} \cong H_s(K(\mathbb{Z},3);\mathbb{F}_p) \otimes V(1)_t \Rightarrow V(1)_{s+t}K(\mathbb{Z},3).$$

From Theorem 2.1 and the formula (2.3) for $V(1)_*$ in low degrees we deduce that $E^2_{*,*}$ consists of $\mathbb{F}_p\{f_0 \cdot v_2,\ e_0 \cdot f_0 \cdot \alpha_1 \cdot \beta_1\}$ in total degree $2p^2 + 2p$. Suspending the relation (2.6) for $k = 1$ we get a relation

$$(P^1)^*(e_1) = e_0. \tag{2.7}$$

Notice that for degree reasons the class $e_1 \cdot f_0 \cdot \beta_1 \in E^2_{*,*}$ survives to $E^{2p-2}_{*,*}$ as a product of e_1 and $f_0 \cdot \beta_1$. By Remark 2.4, and since $f_0 \cdot \beta_1$ is a cycle, we have a differential

$$d^{2p-2}(e_1 \cdot f_0 \cdot \beta_1) = e_0 \cdot f_0 \cdot \alpha_1 \cdot \beta_1,$$

and this implies the claim on $V(1)_{2p^2+2p}K(\mathbb{Z},3)$.

The unit map $S \to K(2)$ factors through a map of ring spectra $V(1) \to K(2)$. The induced ring homomorphism

$$V(1)_*K(\mathbb{Z},2) \to K(2)_*K(\mathbb{Z},2)$$

maps x to $\beta_{(0)}$, since these classes are defined as the double suspension of the unit in $V(1)_0K(\mathbb{Z},0)$, respectively $K(2)_0K(\mathbb{Z},0)$. By construction, the class b' maps to the transpotence of $\beta_{(0)}$, which is $b_{(0,1)}$. We deduce that the sub-$V(1)_*$-algebra of $V(1)_*K(\mathbb{Z},3)$ generated by b' maps surjectively onto the subalgebra

$$P(v_2, b_{(0,1)})/(b_{(0,1)}^p + v_2 b_{(0,1)})$$

of $K(2)_*K(\mathbb{Z},3)$ generated by v_2 and $b_{(0,1)}$. In particular b' is non-nilpotent. Thus $V(1)_{2p^2+2p}K(\mathbb{Z},3)$ is of rank one, and injects into $K(2)_{2p^2+2p}K(\mathbb{Z},3)$. This implies the identity $b'^p = -v_2 b'$.

To prove the Bockstein relation, we map to homology. The Hurewicz homomorphism $h_* : V(1)_*K(\mathbb{Z},3) \to H_*(K(\mathbb{Z},3);\mathbb{F}_p)$ is an isomorphism in degrees 3 and $2p+2$, mapping e_0' to e_0 and b' to the transpotence $\varphi(y) = f_0$ of y. We have a primary homology Bockstein $\beta(f_0) = e_1$ by Theorem 2.1, and combining with (2.7) we obtain $(P^1)^*\beta(f_0) = e_0$. We also have $\beta(P^1)^*(f_0) = 0$ for degree reasons. Finally,

$$Q_1^*(f_0) = \big((P^1)^*\beta - \beta(P^1)^*\big)(f_0) = e_0,$$

so by Lemma 2.3 the relation $\beta_{1,1}(b') = e_0'$ holds. □

3 The units of *ku* and the higher Bott element

The aim of this section is to define low-dimensional classes in $V(1)_*K(ku)$ by using the inclusion of units.

We recall from [30] or [31, Definition 7.6] that the space of units $GL_1(A)$ of an E_∞-ring spectrum A is defined by the following pull-back square of spaces

$$\begin{tikzcd} GL_1(A) \arrow[r] \arrow[d] & \Omega^\infty A \arrow[d, "\pi_0"] \\ GL_1(\pi_0 A) \arrow[r] & \pi_0 A. \end{tikzcd}$$

Taking the vertical fiber over $1 \in GL_1(\pi_0 A)$, we obtain a fiber sequence of group-like E_∞-spaces or infinite loop spaces

$$SL_1(A) \to GL_1(A) \to GL_1(\pi_0 A),$$

with products given by the multiplicative structure of A. Here we can assume that we have a model of $GL_1(A)$ and of $SL_1(A)$ which is actually a topological monoid, see for example [39, Sect. 2.3]. The functor GL_1 from E_∞-ring spectra to infinite loop spaces is right adjoint, up to homotopy, to the suspension functor Σ^∞_+. This follows from [31, Lemma 9.6].

In the case of ku, the space $SL_1(ku)$ is commonly denoted $BU_\otimes$. This notation refers to the product of the underlying H-space of $BU_\otimes$, which represents the tensor product of virtual line bundles.

The first Postnikov section $\pi : BU_\otimes \to K(\mathbb{Z}, 2)$, with homotopy fiber denoted by $BSU_\otimes$, admits a section $j : K(\mathbb{Z}, 2) \simeq BU(1) \to BU_\otimes$. Here the map j represents viewing a line bundle as a virtual line bundle. Both π and j are infinite loop maps, and we have a splitting of infinite loop-spaces

$$BU_\otimes \simeq K(\mathbb{Z}, 2) \times BSU_\otimes,$$

see [30, V.3.1]. We denote by $Bj : K(\mathbb{Z}, 3) \to BBU_\otimes$ a first delooping of j, fitting in a homotopy commutative diagram

$$\begin{tikzcd} K(\mathbb{Z},2) \arrow[r, "j"] \arrow[d, "\tilde{s}"] & BU_\otimes \arrow[d, "\tilde{s}"] \\ \Omega K(\mathbb{Z},3) \arrow[r, "\Omega Bj"] & \Omega BBU_\otimes, \end{tikzcd} \tag{3.1}$$

where $\tilde{s}$ denotes the homotopy equivalence which is right adjoint to the suspension s as in (2.2). We name $y_1 \in \pi_2 K(\mathbb{Z}, 2) \cong \mathbb{Z}$ the generator that maps to $y \in H_2(K(\mathbb{Z}, 2); \mathbb{F}_p)$ by the Hurewicz homomorphism. We have maps of based spaces

$$K(\mathbb{Z}, 2) \xrightarrow{j} BU_\otimes \xrightarrow{c_0} BU \times \{0\} \subset BU \times \mathbb{Z},$$

where c_0 is the inclusion in $BU \times \mathbb{Z}$ followed by the translation of the component of 1 to that of 0 in the H-group $BU \times \mathbb{Z}$. The map $c_0 j$ is a π_2-isomorphism, and we define

$$u = c_{0*} j_*(y_1) \in \pi_2(BU \times \mathbb{Z}).$$

We call u the *Bott class*. We have an isomorphism of rings

$$\pi_*(BU \times \mathbb{Z}) = \pi_* ku \cong \mathbb{Z}[u]$$

given by Bott periodicity. The map $c_{0*} : \pi_*(BU_\otimes) \to \pi_*(BU \times \mathbb{Z})$ is an isomorphism in positive degrees, and we define $y_n \in \pi_{2n}(BU_\otimes)$ by requiring $c_{0*}(y_n) = u^n$. Finally, we define

$$\sigma'_n \in V(1)_{2n+1} BBU_\otimes \tag{3.2}$$

as the image of y_n under the composition

$$\pi_{2n}BU_{\otimes} \xrightarrow{h_*} V(1)_{2n}BU_{\otimes} \xrightarrow{\sigma} V(1)_{2n+1}BBU_{\otimes}.$$

Here the first map is the Hurewicz homomorphism from (unstable) homotopy to $V(1)$-homotopy, and σ is the suspension induced by the map $s : \Sigma BU_{\otimes} \to BBU_{\otimes}$.

Lemma 3.1 *Consider the homomorphism*

$$Bj_* : V(1)_3 K(\mathbb{Z}, 3) \to V(1)_3 BBU_{\otimes}$$

induced by the map defined above. We have $\sigma_1' = (Bj)_*(e_0')$, *where* $e_0' = \sigma^3(1) \in V(1)_3 K(\mathbb{Z}, 3)$, *as given in Definition 2.6.*

Proof We have a commutative diagram

$$\begin{array}{ccccc}
\pi_2 K(\mathbb{Z},2) & \xrightarrow{h_*} & V(1)_2 K(\mathbb{Z},2) & \xrightarrow{\sigma} & V(1)_3 K(\mathbb{Z},3) \\
\downarrow j_* & & \downarrow j_* & & \downarrow Bj_* \\
\pi_2 BU_{\otimes} & \xrightarrow{h_*} & V(1)_2 BU_{\otimes} & \xrightarrow{\sigma} & V(1)_3 BBU_{\otimes}.
\end{array}$$

The right-hand square is induced in $V(1)$-homotopy from the square left adjoint to the square (3.1). The class $y_1 \in \pi_2 K(\mathbb{Z}, 2)$ was chosen so that $h_*(y_1) = \sigma^2(1)$ in $V(1)_2 K(\mathbb{Z}, 2) \cong H_2(K(\mathbb{Z}, 2); \mathbb{F}_p)$. The lemma follows, since

$$\sigma_1' = \sigma h_* j_*(y_1) = (Bj)_* \sigma h_*(y_1) = (Bj)_* \sigma^3(1) = (Bj)_*(e_0'). \qquad \square$$

The space $\Omega^\infty K(ku)$ is defined as the group completion of the topological monoid $\coprod_n BGL_n(ku)$, with product modelling the block-sum of matrices, see for instance [20, VI.7]. The composition

$$w : BBU_{\otimes} \to BGL_1(ku) \to \coprod_n BGL_n(ku) \to \Omega^\infty K(ku) \tag{3.3}$$

factors through an infinite loop map $BBU_{\otimes} \to SL_1 K(ku)$, which is right adjoint to a map

$$\omega : \Sigma^\infty_+ BBU_{\otimes} \to K(ku)$$

of commutative S-algebras. We consider also the map of commutative S-algebras

$$\phi : \Sigma^\infty_+ K(\mathbb{Z}, 3) \to K(ku)$$

defined as the composition of the suspension of $Bj : K(\mathbb{Z},3) \to BBU_{\otimes}$ with the map ω.

Definition 3.2 For $n \geq 1$, we define

$$\sigma_n = \omega_*(\sigma_n') \in V(1)_{2n+1}K(ku),$$

where σ_n' is the class given in (3.2). We define the "higher Bott element" as

$$b = \phi_*(b') \in V(1)_{2p+2}K(ku),$$

where $b' \in V(1)_{2p+2}K(\mathbb{Z},3)$ is the class given in Definition 2.6.

Remark 3.3 Notice that by Proposition 2.7 the classes b and σ_1 are related by a primary mod (v_1) homotopy Bockstein $\beta_{1,1}(b) = \sigma_1$.

Remark 3.4 Assume that p is an odd prime. If R is a number ring containing a primitive p-th root of unity ζ_p, for example $R = \mathbb{Z}[\zeta_p]$, then the mod (p) algebraic K-theory of R contains a non-nilpotent class

$$\beta \in V(0)_2K(R),$$

called the Bott element, which we referred to in the introduction. It was defined by Browder [15] using the composition

$$BC_p \to BGL_1R \to \Omega^\infty K(R)$$

analogous to (3.3), and its adjoint

$$\phi : \Sigma_+^\infty BC_p \to K(R).$$

Here C_p denotes the cyclic subgroup of order p of $GL_1(R)$ generated by ζ_p. By inspection, the class $x = \zeta_p - 1$ satisfies $x^p = 0$ in the group-ring $\mathbb{F}_p[C_p] = V(0)_0C_p$, and has a well defined "transpotence" $\beta' \in V(0)_2BC_p$, supporting a primary mod (p) homotopy Bockstein $\beta_{0,1}(\beta') \doteq \sigma(1) \in V(0)_1BC_p$. The classical Bott element can then be defined as

$$\beta = \phi_*(\beta') \in V(0)_2K(R).$$

An embedding of rings $R \subset \mathbb{C}^{\text{top}}$, where $\mathbb{C}^{\text{top}}$ has the Euclidean topology, induces a map of commutative S-algebras $\iota : K(R) \to K(\mathbb{C}^{\text{top}}) = ku$ in algebraic K-theory. Browder's Proposition [15, 2.2] implies that $\iota_*\phi_*(\beta') = u$, where u is the Bott class in $V(0)_*ku \cong P(u)$. This proves that β is non-nilpotent and is related to the Bott periodicity of topological K-theory. Snaith showed [40] that the relation $\beta'^p = v_1\beta'$ in $V(0)_*BC_p$ promotes to the relation

$$\beta^{p-1} = v_1$$

in $V(0)_*K(R)$.

The remark above makes it clear that our construction of $b \in V(1)_{2p+2}K(ku)$ is inspired from the classical Bott element, and that these classes share interesting properties. This provides some justification for calling b a *higher Bott element*. Here *higher* refers to the fact that b lives one chromatic step higher than β, in the sense that it is defined only in algebraic K-theory modulo (p, v_1) and that it is related to v_2-periodicity. Indeed, recall from Theorem 1.1 and Proposition 1.3 that b is non-nilpotent and that the relation $b'^p = -v_2b'$ in $V(1)_*K(\mathbb{Z}, 3)$ promotes to the relation

$$b^{p-1} = -v_2$$

in $V(1)_*K(ku)$. Our proof of these assertions relies on the computation of the cyclotomic trace for ku, and is much more technical then in the number ring case: unfortunately, in the present situation we don't have an analogue of the map $K(R) \to K(\mathbb{C}^{\mathrm{top}})$, but see the remark below for a possible candidate.

Remark 3.5 John Rognes conjectured [4] that if Ω_1 is a separably closed $K(1)$-local pro-Galois extension of ku, in the sense of [38], then there is a weak equivalence

$$L_{K(2)}K(\Omega_1) \simeq E_2,$$

where $L_{K(2)}$ is the Bousfield localization functor with respect to the Morava K-theory $K(2)$, and where E_2 is the second Morava E-theory spectrum [21] with coefficients

$$(E_2)_* = W(\mathbb{F}_{p^2})[[u_1]][u, u^{-1}].$$

This would provide a map

$$\iota : K(ku) \to L_{K(2)}K(\Omega_1) \simeq E_2$$

that might play the role, at this chromatic level, of the map $K(R) \to K(\mathbb{C}^{\mathrm{top}})$ mentioned in Remark 3.4. Since $V(1)_*E_2 \cong \mathbb{F}_{p^2}[u, u^{-1}]$ with $u^{p^2-1} = v_2$, we presume that the class b would be detected by the non-nilpotent class

$$\iota_*(b) = \alpha u^{p+1} \in V(1)_*E_2$$

for some $\alpha \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$ with $\alpha^{p-1} = -1$. More generally, we expect that a periodic higher Bott element can be defined in $V(1)_*K(A)$ if A is an commutative S-algebra with an S-algebra map $A \to \Omega_1$ and a suitable $(p-1)$th-root of v_1 in $V(0)_*A$.

4 The trace map

In this section, we consider the Bökstedt trace map [11]

$$\mathrm{tr} : K(ku) \to THH(ku)$$

to topological Hochschild homology. This is a map of commutative S-algebras, and it induces a homomorphism of graded-commutative algebras in $V(1)$-homotopy, which we just call the trace. Our aim here is to prove that for $n \le p-2$ the classes σ_n and b defined above are non-zero in $V(1)_*K(ku)$, as well as some of their products, see Proposition 4.6. We achieve this by showing that these classes have a non-zero trace in $V(1)_*THH(ku)$. To this end, we briefly recall the computation of $V(1)_*THH(ku)$ given in [1, 9.15].

The topological Hochschild homology spectrum $THH(ku)$ is a ku-algebra, and its $V(1)$-homotopy groups form an algebra over the truncated polynomial algebra $V(1)_*ku = P_{p-1}(u)$, where we also denote by u the mod (p, v_1) reduction of the Bott class $u \in \pi_2 ku$. There is a free $\mathbb{F}_p$-sub-algebra $E(\lambda_1) \otimes P(\mu)$ in $V(1)_*THH(ku)$, and there is an isomorphism of $E(\lambda_1) \otimes P(\mu) \otimes P_{p-1}(u)$-modules

$$V(1)_*THH(ku) \cong E(\lambda_1) \otimes P(\mu) \otimes Q_*, \tag{4.1}$$

where Q_* is the $P_{p-1}(u)$-module given by

$$Q_* = P_{p-1}(u) \oplus P_{p-2}(u)\{a_0, b_1, a_1, b_2, \dots, a_{p-2}, b_{p-1}\} \oplus P_{p-1}(u)\{a_{p-1}\}.$$

The degree of these generators is given by $|\lambda_1| = 2p-1$, $|\mu| = 2p^2$, $|a_i| = 2pi+3$ and $|b_j| = 2pj+2$. The isomorphism (4.1) is an isomorphism of $P_{p-1}(u)$-algebras if the product on the $P_{p-1}(u)$-module generators of Q_* is given by the relations

$$\begin{cases} b_i b_j = u b_{i+j} & i+j \le p-1, \\ b_i b_j = u b_{i+j-p}\mu & i+j \ge p, \\ a_i b_j = u a_{i+j} & i+j \le p-1, \\ a_i b_j = u a_{i+j-p}\mu & i+j \ge p, \\ a_i a_j = 0 & 0 \le i, j \le p-1. \end{cases} \tag{4.2}$$

Here by convention $b_0 = u$. For example we have a product

$$(u^k a_i)(u^l b_j) = u^{p-2} a_{p-1}$$

if $k+l = p-3$ and $i+j = p-1$.

Remark 4.1 The class μ is called μ_2 in [1], but we adopt here the notation of [2].

The classes $u^{n-1}a_0 \in V(1)_{2n+1}THH(ku)$ for $1 \leq n \leq p-2$ are constructed as follows. The circle action $S^1_+ \wedge THH(ku) \to THH(ku)$ restricts in the homotopy category to a map $d : \Sigma THH(ku) \to THH(ku)$, which in any homology theory E_* induces Connes' operator

$$d : E_*THH(ku) \to E_{*+1}THH(ku). \tag{4.3}$$

We have an S-algebra map $l : ku \to THH(ku)$ given by the inclusion of zero-simplices. Composing the induced map in E_*-homology with d yields a suspension homomorphism

$$dl_* : E_*ku \to E_{*+1}THH(ku),$$

see [32, 3.2] (it is often denoted σ). For $1 \leq n \leq p-2$, we define the class $u^{n-1}a_0$ as the image

$$u^{n-1}a_0 = dl_*(u^n)$$

of $u^n \in V(1)_*ku$. Mapping to homology, we can show that these classes are non-zero. By Lemma 2.3, the Hurewicz homomorphism

$$h_* : V(1)_*THH(ku) \to H_*(THH(ku); \mathbb{F}_p)$$

is an isomorphism in degrees $* \leq 2p-3$ (notice that $\alpha_1 = 0$ in $V(1)_*THH(ku)$ since $THH(ku)$ is a ku-algebra). Let $x = h_*(u) \in H_2(ku; \mathbb{F}_p)$ be the image of $u \in V(1)_2ku$. We then have $h_*(u^{n-1}a_0) = dl_*(x^n)$ in $H_{2n+1}(THH(ku); \mathbb{F}_p)$, and this class represents the permanent cycle $1 \otimes x^n \in E^1_{1,2n}(ku)$ in the Bökstedt spectral sequence

$$E^1_{s,*}(ku) = H_*(ku; \mathbb{F}_p)^{\otimes(s+1)},$$
$$E^2_{s,*}(ku) = HH^{\mathbb{F}_p}_{s,*}\big(H_*(ku; \mathbb{F}_p)\big) \Rightarrow H_{s+*}(THH(ku); \mathbb{F}_p).$$

This proves that the classes $h_*(u^{n-1}a_0)$ are non-zero for these values of n. We refer to [1, Sect. 9] for more details.

Lemma 4.2 *If* $1 \leq n \leq p-2$, *the class* σ'_n *of* (3.2) *maps to the class* $u^{n-1}a_0$ *under the composition*

$$V(1)_*BBU_\otimes \xrightarrow{\omega_*} V(1)_*K(ku) \xrightarrow{\mathrm{tr}_*} V(1)_*THH(ku).$$

Proof As mentioned above, $h_* : V(1)_{2n+1}THH(ku) \to H_{2n+1}(THH(ku); \mathbb{F}_p)$ is an isomorphism for $n \leq p-2$ and maps $u^{n-1}a_0$ to $dl_*(x^n)$. Thus, passing to homology and using the definition of σ'_n in (3.2), if suffices to prove that the composition

$$H_{2n}(BU_\otimes; \mathbb{F}_p) \xrightarrow{\sigma} H_{2n+1}(BBU_\otimes; \mathbb{F}_p) \xrightarrow{\mathrm{tr}_*\omega_*} H_{2n+1}(THH(ku); \mathbb{F}_p)$$

maps $z_n = h_*(y_n) \in H_{2n}(BU_\otimes; \mathbb{F}_p)$ to $dl_*(x^n)$. Here we also denoted by h_* the Hurewicz homomorphism $\pi_{2n}BU_\otimes \to H_{2n}(BU_\otimes; \mathbb{F}_p)$. First, we need some information on the trace map. We will use the following commutative diagram of spaces

$$\begin{tikzcd} BBU_\otimes \arrow[r, "i"] \arrow[d, "w"] & B^{\mathrm{cy}}BU_\otimes \arrow[d, "\tau"] & BU_\otimes \arrow[l, "l"'] \arrow[d, "c_1"] \\ \Omega^\infty K(ku) \arrow[r, "\Omega^\infty \mathrm{tr}"] & \Omega^\infty THH(ku) & \Omega^\infty ku, \arrow[l, "l"'] \end{tikzcd} \tag{4.4}$$

which is assembled from [39, Sect. 4]. Here the space $B^{\mathrm{cy}}BU_\otimes$ is the realization of the cyclic nerve of the topological monoid $BU_\otimes$ and, as $\Omega^\infty THH(ku)$, is equipped with a canonical S^1-action. The map τ is the realization of a morphism of cyclic spaces, and is therefore S^1-equivariant. The maps l are given by the inclusion of 0-simplices, while c_1 is the inclusion of the component of 1. There is a homotopy fibration [39, Proposition 3.1]

$$BU_\otimes \xrightarrow{l} B^{\mathrm{cy}}BU_\otimes \xrightarrow{p} BBU_\otimes, \tag{4.5}$$

and the map p admits a section up to homotopy $i : BBU_\otimes \to B^{\mathrm{cy}}BU_\otimes$.

Let d be Connes' operator on $H_*(B^{\mathrm{cy}}BU_\otimes; \mathbb{F}_p)$ and $H_*(\Omega^\infty THH(ku); \mathbb{F}_p)$. It commutes with $\tau_* : H_*(B^{\mathrm{cy}}BU_\otimes; \mathbb{F}_p) \to H_*(\Omega^\infty THH(ku); \mathbb{F}_p)$ since τ is equivariant. In the next lemma, we prove that

$$dl_*(z_n) = i_*\sigma(z_n)$$

holds in $H_{2n+1}(B^{\mathrm{cy}}BU_\otimes; \mathbb{F}_p)$. Using (4.4), we deduce

$$(\Omega^\infty \mathrm{tr})_* w_*\sigma(z_n) = \tau_* i_*\sigma(z_n) = \tau_* dl_*(z_n) = d\tau_* l_*(z_n) = dl_* c_{1*}(z_n).$$

Finally, composing with the stabilization map

$$\mathrm{st} : H_*(\Omega^\infty THH(ku); \mathbb{F}_p) \to H_*(THH(ku); \mathbb{F}_p)$$

to spectrum homology, we obtain

$$\mathrm{tr}_*\omega_*\sigma(z_n) = \mathrm{st}(\Omega^\infty \mathrm{tr})_* w_*\sigma(z_n) = \mathrm{st}dl_* c_{1*}(z_n) = dl_*(x^n).$$

For the last equality, we used that the stabilization commutes with dl_*, and that $\mathrm{st}c_{1*}(z_n) = x^n$ for $1 \le n \le p-2$. $\square$

Lemma 4.3 *The equality $dl_*(z_n) = i_*\sigma(z_n)$ holds in $H_{2n+1}(B^{\mathrm{cy}}BU_\otimes; \mathbb{F}_p)$.*

Proof We consider the homotopy fibration (4.5). Since $H_*(BU_\otimes; \mathbb{F}_p)$ is concentrated in even degrees, the map $p_* : H_*(B^{\mathrm{cy}}BU_\otimes; \mathbb{F}_p) \to H_*(BBU_\otimes; \mathbb{F}_p)$ restricts to an isomorphism

$$p_* : \mathrm{Prim}\big(H_{2n+1}(B^{\mathrm{cy}}BU_\otimes; \mathbb{F}_p)\big) \to \mathrm{Prim}\big(H_{2n+1}(BBU_\otimes; \mathbb{F}_p)\big)$$

of the subgroups of primitive elements in degree $2n+1$, with the restriction of i_* as inverse. The class $l_*(z_n)$ is spherical, hence primitive, and it follows from $d(1) = 0$ that $dl_*(z_n)$ is also primitive.

Next, we consider the diagram

$$\begin{tikzcd}
S^1 \times BU_\otimes \arrow[r, "1\times l"] \arrow[d] & S^1 \times B^{\mathrm{cy}}BU_\otimes \arrow[r, "\mu"] & B^{\mathrm{cy}}BU_\otimes \arrow[d, "p"] \\
S^1 \wedge BU_\otimes \arrow[rr, "s"] & & BBU_\otimes,
\end{tikzcd}$$

where μ denotes the S^1-action on $B^{\mathrm{cy}}BU_\otimes$ and s the suspension map (2.2). This diagram is commutative, as can be checked at simplicial level by using the definition of μ, see for example [27, 7.1.9]. Therefore $p_*dl_*(z_n) = \sigma(z_n)$, and since $dl_*(z_n)$ is primitive, we have

$$dl_*(z_n) = i_*p_*dl_*(z_n) = i_*\sigma(z_n). \qquad \square$$

Lemma 4.4 *The class b' maps to the class b_1 under the composition*

$$V(1)_*K(\mathbb{Z}, 3) \xrightarrow{\phi_*} V(1)_*K(ku) \xrightarrow{\mathrm{tr}_*} V(1)_*THH(ku).$$

Proof We know from Lemmas 3.1 and 4.2 that $e_0' \in V(1)_3K(\mathbb{Z}, 3)$ maps to the class a_0 in $V(1)_3THH(ku)$. We have primary mod (v_1) homotopy Bockstein

$$\beta_{1,1}(b') = e_0' \quad \text{and} \quad \beta_{1,1}(b_1) = a_0$$

in $V(1)_*K(\mathbb{Z}, 3)$ and $V(1)_*THH(ku)$ respectively, see Proposition 2.7 and [1, 9.19]. Moreover $V(1)_{2p+2}THH(ku) = \mathbb{F}_p\{b_1\}$, so that $\beta_{1,1}$ is injective on this group. The result follows, since

$$\beta_{1,1}\mathrm{tr}_*\phi_*(b') = \mathrm{tr}_*\phi_*\beta_{1,1}(b') = \mathrm{tr}_*\phi_*(e_0') = a_0. \qquad \square$$

Let $\kappa : ku \to ku_p$ be the completion at p. It induces the inclusion $\mathbb{Z}[u] \to \mathbb{Z}_p[u]$ of coefficients rings.

Definition 4.5 We also denote by

$$\sigma_n \in V(1)_{2n+1}K(ku_p) \text{ and } b \in V(1)_{2p+2}K(ku_p)$$

the image under $\kappa_* : V(1)_*K(ku) \to V(1)_*K(ku_p)$ of the classes σ_n and b defined in Definition 3.2.

Proposition 4.6 *The classes*

$$\begin{cases} b^k & \text{for } 0 \le k \le p-2, \text{ and} \\ \sigma_n b^l & \text{for } 1 \le n \le p-2 \text{ and } 0 \le l \le p-2-n \end{cases}$$

are non-zero in $V(1)_*K(ku)$ and in $V(1)_*K(ku_p)$.

Proof For $V(1)_*K(ku)$, it follows from Lemmas 4.2, 4.4 and the structure of $V(1)_*THH(ku)$ given in (4.2). In more detail, we have $\mathrm{tr}_*(b^k) = b_1^k \neq 0$ for $k \le p-2$ and $\mathrm{tr}_*(\sigma_n b^l) = u^{n-1}a_0 b_1^l = u^{n+l-1}a_l \neq 0$ for $l \le p-3$ and $n+l-1 \le p-3$. Notice that we have a commutative diagram

$$\begin{tikzcd} V(1)_*K(ku) \arrow[r, "\mathrm{tr}_*"] \arrow[d, "\kappa_*"'] & V(1)_*THH(ku) \arrow[d, "\kappa_*"] \\ V(1)_*K(ku_p) \arrow[r, "\mathrm{tr}_*"] & V(1)_*THH(ku_p). \end{tikzcd}$$

The map $\kappa : THH(ku) \to THH(ku_p)$ is a weak equivalence after p-completion, so in this diagram the right-hand κ_* is an isomorphism. This proves that the result also holds for $V(1)_*K(ku_p)$. $\square$

Remark 4.7 We claimed in Theorem 1.1 and Proposition 1.3 that b is non-nilpotent in $V(1)_*K(ku)$. However, we have

$$\mathrm{tr}_*(b^{p-1}) = \mathrm{tr}_*(b)^{p-1} = b_1^{p-1} = u^{p-2}b_{p-1} = 0$$

in $V(1)_*THH(ku)$, so that the Bökstedt trace is not sufficient for proving this assertion. This is of course also predicted by our other claim that $b^{p-1} = -v_2$ holds in $V(1)_*K(ku)$. Indeed, v_2 maps to zero in $V(1)_*THH(ku)$ since $THH(ku)$ is a ku-algebra.

5 Algebraic K-theory in low degrees

In this section, we compute the groups $V(1)_*K(ku_p)$ in degrees $* \le 2p-2$. This complements the computations presented in the next sections, which are based on evaluating the fixed points of $THH(ku)$ and which are valid only in degrees larger than $2p-2$, see Proposition 6.7.

Consider the Adams summand

$$\ell_p = ku_p^{h\Delta}$$

of ku_p, where $\Delta \cong \mathbb{Z}/(p-1)$ is the finite subgroup of the p-adic units, acting on ku_p by p-adic Adams operations, and where $(-)^{h\Delta}$ denotes the homotopy fixed points. By Theorem 10.2 of [1], the natural map $V(1)_*K(\ell_p) \to V(1)_*K(ku_p)$ factors through an isomorphism

$$V(1)_*K(\ell_p) \cong \big(V(1)_*K(ku_p)\big)^{\Delta} \subset V(1)_*K(ku_p) \tag{5.1}$$

onto the elements of $V(1)_*K(ku_p)$ fixed under the induced action of Δ. In the sequel, we identify $V(1)_*K(\ell_p)$ with its image in $V(1)_*K(ku_p)$.

The $V(1)$-homotopy of $K(\ell_p)$ is computed in [2]. In the degrees we are concerned with here, namely $* \le 2p-2$, $V(1)_*K(\ell_p)$ is generated as an $\mathbb{F}_p$-vector space by the classes listed in

$$\{1, \lambda_1 t^d, s, \partial\lambda_1 \,|\, 0 < d < p\}, \tag{5.2}$$

of degree $|\lambda_1 t^d| = 2p-2d-1$, $|s| = 2p-3$ and $|\partial\lambda_1| = 2p-2$, see [2, 9.1] (where the sporadic v_2-torsion class s was denoted a). The zeroth Postnikov section $\ell_p \to H\mathbb{Z}_p$ is a $(2p-2)$-connected map, so that the induced map $K(\ell_p) \to K(\mathbb{Z}_p)$ is $(2p-1)$-connected [9, Proposition 10.9]. All the classes listed in (5.2) map to classes with same name in $V(1)_*K(\mathbb{Z}_p)$, which is given by the formula

$$V(1)_*K(\mathbb{Z}_p) \cong E(\lambda_1) \oplus \mathbb{F}_p\{s, \partial\lambda_1\} \oplus \mathbb{F}_p\{\lambda_1 t^d \,|\, 0 < d < p\}.$$

The name of the classes in this formula refers to permanent cycles in the S^1 homotopy fixed-point spectral sequence used in the computation of $V(1)_*K(\mathbb{Z}_p)$ by traces, compare with Theorem 7.9. If desired, these classes could be given a more memorable name by means of the inclusion

$$V(0)_*K(\mathbb{Z}_p) \to V(0)_*K(\mathbb{Q}_p(\zeta_p)),$$

in the target of which they can be decomposed as a product of a unit and a power of the Bott element $\beta \in V(0)_2K(\mathbb{Q}_p(\zeta_p))$.

Using the inclusion given in (5.1), we view the classes listed in (5.2) as elements of $V(1)_*K(ku_p)$. The following lemma implies that these classes are linearly independent of the classes in $V(1)_*K(ku_p)$ constructed in the previous section.

Lemma 5.1 *The non-zero classes b^k and $\sigma_n b^l$ in $V(1)_*K(ku_p)$ given in Proposition 4.6 are not fixed under the action of Δ.*

Proof All these classes map into $V(1)_*THH(ku)$ to classes which do not lie in the image of $V(1)_*THH(\ell_p)$, and hence which are not fixed under the action of Δ, see Proposition 10.1 of [1]. □

Proposition 5.2 *The inclusion*

$$\mathbb{F}_p\{1, \sigma_n, \lambda_1 t^d, s, \partial\lambda_1 \mid 1 \leq n \leq p-2,\ 0 < d < p\} \subset V(1)_*K(ku_p)$$

of graded $\mathbb{F}_p$-vector spaces is an isomorphism in degrees $\leq 2p-2$.

Proof We have constructed all the classes listed above and have argued that they are linearly independent. It suffices therefore to compute the dimension of $V(1)_n K(ku_p)$ as an $\mathbb{F}_p$-vector space for all $0 \leq n \leq 2p-2$.

Consider a double loop map $\Omega S^3 \to BU_\otimes$ such that the composition

$$S^2 \to \Omega S^3 \to BU_\otimes,$$

where $S^2 \to \Omega S^3$ is the adjunction unit, represents the class $y_1 \in \pi_2 BU_\otimes$ defined in Sect. 3. By adjunction we have a map of E_2-ring spectra

$$S[\Omega S^3] \to ku,$$

where $S[\Omega S^3]$ is another notation for the suspension spectrum $\Sigma^\infty_+ \Omega S^3$. We refer to [3, Proposition 2.2] for some more details on the construction of this map. After p-completion this map is $(2p-3)$-connected, and induces a $(2p-2)$-connected map $K(S[\Omega S^3]_p) \to K(ku_p)$. The dimension of the $\mathbb{F}_p$-vector space $V(1)_n K(S[\Omega S^3]_p)$ for $n \leq 2p-2$ is computed in the following lemma, and this completes the proof of this proposition. Notice that a priory

$$V(1)_{2p-2} K(S[\Omega S^3]_p) \to V(1)_{2p-2} K(ku_p)$$

is only surjective, but luckily $V(1)_{2p-2} K(S[\Omega S^3]_p)$ is of rank one. Since we know that the rank of $V(1)_{2p-2} K(ku_p)$ is at least one, we also have an isomorphism in this degree. □

Lemma 5.3 *The dimension of $V(1)_n K(S[\Omega S^3]_p)$ as an $\mathbb{F}_p$-vector space is*

$$\begin{cases} 1 & \text{if } n = 0, 1, 2p-2, \\ 2 & \text{if } n \text{ is odd with } 3 \le n \le 2p-5, \\ 3 & \text{if } n = 2p-3, \\ 0 & \text{for other values of } n \le 2p-2. \end{cases}$$

Proof We compute $V(1)_* K(S[\Omega S^3]_p)$ in degrees less than $2p-1$ by using the cyclotomic trace map to topological cyclic homology [11], which sits in a cofibre sequence [22]

$$K(S[\Omega S^3]_p)_p \xrightarrow{\text{trc}} TC(S[\Omega S^3]_p) \to \Sigma^{-1} H\mathbb{Z}_p \to \Sigma K(S[\Omega S^3]_p)_p.$$

Here $TC(X) = TC(X; p)$ denotes the (p-completed) topological cyclic homology spectrum of a spectrum X. By inspection, it suffices to prove that we have

$$\dim_{\mathbb{F}_p} V(1)_n TC(S[\Omega S^3]_p) = \begin{cases} 1 & \text{if } n = -1, 0, 1, 2p-2, \\ 2 & \text{if } n \text{ is odd with } 3 \le n \le 2p-3, \\ 0 & \text{for other values of } n \le 2p-2. \end{cases} \quad (5.3)$$

Indeed, $V(1)_* \Sigma^{-1} H\mathbb{Z}_p$ consists of a copy of $\mathbb{F}_p$ in degrees -1 and $2p-2$, and is zero in other degrees. We have an isomorphism $V(1)_{-1} TC(S[\Omega S^3]_p) \to V(1)_{-1} \Sigma^{-1} H\mathbb{Z}_p$, and the sporadic class s is in the image of the connecting homomorphism

$$V(1)_{2p-2} \Sigma^{-1} H\mathbb{Z}_p \to V(1)_{2p-3} K(S[\Omega S^3]_p),$$

by naturality with respect to $S[\Omega S^3]_p \to H\mathbb{Z}_p$, see for example [2, Proof of 9.1].

The reduced topological cyclic homology spectrum $\widetilde{TC}(S[\Omega S^3]_p)$ is the homotopy fibre of the map $c : TC(S[\Omega S^3]_p) \to TC(S_p)$ induced by the map $S^3 \to *$ to a one-point space. The maps c admits a splitting, and we have a decomposition

$$TC(S[\Omega S^3]_p) \simeq TC(S_p) \vee \widetilde{TC}(S[\Omega S^3]_p).$$

The spectrum $TC(S_p)$ decomposes as

$$TC(S_p) \simeq S_p \vee \Sigma \mathbb{C}P^{\infty}_{-1},$$

where $\mathbb{C}P^{\infty}_{-1}$ is the (p-completed) Thom spectrum of minus the canonical line bundle on $\mathbb{C}P^{\infty}$, see [29]. The homology of $\Sigma \mathbb{C}P^{\infty}_{-1}$ is given by

$$H_*(\Sigma \mathbb{C}P^{\infty}_{-1}; \mathbb{F}_p) \cong \mathbb{F}_p\{x_i \mid i \ge -1\}$$

with $|x_i| = 2i + 1$. Moreover these classes can be chosen so that the relations

$$(P^1)^*(x_{p-2}) = x_{-1} \text{ and } (P^1)^*(x_{p-1}) = 0$$

hold. It follows from Lemma 2.3 that we have an inclusion

$$\mathbb{F}_p\{c_i \mid -1 \le i \le p-3\} \cup \mathbb{F}_p\{\alpha(x_0)\} \subset V(1)_*(\Sigma \mathbb{C}P^{\infty}_{-1}),$$

which is an isomorphism in degrees $* \le 2p-2$, with $h_*(c_i) = x_i$. These classes have degree $|c_i| = 2i+1$ and $|\alpha(x_0)| = 2p-2$.

By [12, 3.9], we have a decomposition

$$\widetilde{TC}(S[\Omega S^3]_p) \simeq \Sigma^\infty S^3_p \vee \widetilde{V},$$

where $\widetilde{V}$ is the (p-completed) homotopy fiber of the composition

$$\Sigma^\infty \Sigma(ES^1_+ \wedge_{S^1} LS^3) \xrightarrow{\text{trf}} \Sigma^\infty LS^3 \xrightarrow{\epsilon_1} \Sigma^\infty S^3.$$

Here trf is the dimension-shifting S^1-transfer on the free loop space LS^3 of S^3, and ϵ_1 is the evaluation at $1 \in S^1$, see [29]. We consider the Serre spectral sequence

$$E^2_{**} = H_*\big(BS^1; H_*(LS^3, \mathbb{F}_p)\big) \Rightarrow H_*(ES^1 \times_{S^1} LS^3; \mathbb{F}_p).$$

We have isomorphisms

$$\begin{aligned} &H_*(BS^1; \mathbb{F}_p) \cong H_*(K(\mathbb{Z}, 2); \mathbb{F}_p) = \Gamma(y) \quad \text{and} \\ &H_*(LS^3; \mathbb{F}_p) \cong P(z) \otimes E(dz). \end{aligned}$$

Here $z \in H_2(\Omega S^3; \mathbb{F}_p) \subset H_2(LS^3; \mathbb{F}_p)$ and $dz \in H_3(LS^3; \mathbb{F}_p)$ is the suspension of z associated to the circle action on LS^3. In particular, we have a non-zero d^2-differential

$$d^2(yz) = dz.$$

For degree reasons no further non-zero differential involves the classes in total degree less than $2p$, and we have an inclusion

$$P_p(y) \oplus \mathbb{F}_p\{z^j \mid 1 \le j \le p-1\} \subset H_*(ES^1 \times_{S^1} LS^3; \mathbb{F}_p)$$

which is an isomorphism in degrees less than $2p$. We deduce that the inclusion

$$\Sigma \mathbb{F}_p\{z^j \mid 1 \le j \le p-1\} \subset H_*(\Sigma^\infty \Sigma(ES^1_+ \wedge_{S^1} LS^3); \mathbb{F}_p)$$

is an isomorphism in degrees less the $2p-1$. The homomorphism

$$(\epsilon_1 \mathrm{trf})_* : H_*(\Sigma^\infty \Sigma(ES^1_+ \wedge_{S^1} LS^3); \mathbb{F}_p) \to H_*(S^3; \mathbb{F}_p) = E(e)$$

maps Σz to a generator e of $H_3(S^3; \mathbb{F}_p)$ since the restriction of trf to $\Sigma^\infty \Sigma(S^1_+ \wedge_{S^1} LS^3)$ is induced by the circle action. This implies that we have an inclusion

$$\mathbb{F}_p\{e, \Sigma z^j \,|\, 2 \le j \le p-2\} \subset H_*(\Sigma^\infty S^3_p \vee \widetilde{V}; \mathbb{F}_p) \cong H_*(\widetilde{TC}(S[\Omega S^3]_p); \mathbb{F}_p)$$

which is an isomorphism in degrees smaller than $2p-1$. By Lemma 2.3

$$\mathbb{F}_p\{e, \Sigma z^j \,|\, 2 \le j \le p-2\} \subset V(1)_* \widetilde{TC}(S[\Omega S^3]_p)$$

is also an isomorphism in degrees less than $2p-1$. In summary, we have

$$V(1)_* TC(S[\Omega S^3]_p) \cong V(1)_* \oplus V(1)_* \Sigma \mathbb{C}P^\infty_{-1} \oplus V(1)_* \widetilde{TC}(S[\Omega S^3]_p),$$

which is isomorphic to

$$\mathbb{F}_p\{1, \alpha_1, c_i, \alpha(x_0), e, \Sigma z^j \,|\, -1 \le i \le p-3,\ 2 \le j \le p-2\}$$

in degrees smaller than $2p-1$. This proves that formula (5.3) for the rank of the $\mathbb{F}_p$-vector space $V(1)_* TC(S[\Omega S^3]_p)$ is correct. □

Remark 5.4 In an earlier proof of this lemma we used the space $BU(1)$ and the map $\theta : \Sigma^\infty_+ BU(1) \to ku$ of commutative S-algebras. I thank John Rognes for noticing that using ΩS^3 instead simplifies the computation. The maps

$$S[\Omega S^3] \to \Sigma^\infty_+ BU(1) \to ku$$

are π_0-isomorphisms and rational equivalences. We use this in [3] to determine the rational homotopy type of $K(ku)$.

6 The fixed points

In this section we compute the $V(1)$-homotopy groups of the homotopy limit

$$TF(ku_p) = \mathrm{holim}_{n,F}\, THH(ku_p)^{C_{p^n}},$$

where $F : THH(ku_p)^{C_{p^{n+1}}} \to THH(ku_p)^{C_{p^n}}$ is the Frobenius map. This will be used in the next section to compute the topological cyclic homology of ku_p. The strategy to perform such computations was developed in [9, 22, 43], but we will closely follow the exposition and adopt the notations of [2, Sects. 3, 5 and 6], with an exception: the G Tate construction on an equivariant G spectrum X will be denoted by X^{tG} instead of $\hat{\mathbb{H}}(G, X)$. We refer the reader to [2, Sect. 3] for a brief review of the homotopy commutative norm-restriction diagram

$$\begin{tikzcd}
 & K(ku_p) \arrow[d, "\mathrm{tr}_n"'] \arrow[dr, "\mathrm{tr}_{n-1}"] & & \\
THH(ku_p)_{hC_{p^n}} \arrow[r, "N"] \arrow[d, equal] & THH(ku_p)^{C_{p^n}} \arrow[r, "R"] \arrow[d, "\Gamma_n"] & THH(ku_p)^{C_{p^{n-1}}} \arrow[r] \arrow[d, "\hat{\Gamma}_n"] & \Sigma THH(ku_p)_{hC_{p^n}} \arrow[d, equal] \\
THH(ku_p)_{hC_{p^n}} \arrow[r, "N^h"] & THH(ku_p)^{hC_{p^n}} \arrow[r, "R^h"] & THH(ku_p)^{tC_{p^n}} \arrow[r] & \Sigma THH(ku_p)_{hC_{p^n}}
\end{tikzcd}$$

for any $n \geq 1$, which is our essential tool. By passage to homotopy limits over the Frobenius maps, we obtain the homotopy commutative diagram

$$\begin{tikzcd}
 & K(ku_p) \arrow[d, "\mathrm{tr}_F"'] \arrow[dr, "\mathrm{tr}_F"] & & \\
\Sigma THH(ku_p)_{hS^1} \arrow[r, "N"] \arrow[d, equal] & TF(ku_p) \arrow[r, "R"] \arrow[d, "\Gamma"] & TF(ku_p) \arrow[r] \arrow[d, "\hat{\Gamma}"] & \Sigma^2 THH(ku_p)_{hS^1} \arrow[d, equal] \\
\Sigma THH(ku_p)_{hS^1} \arrow[r, "N^h"] & THH(ku_p)^{hS^1} \arrow[r, "R^h"] & THH(ku_p)^{tS^1} \arrow[r] & \Sigma^2 THH(ku_p)_{hS^1}.
\end{tikzcd}$$

The map $i_* : V(1)_* THH(\ell_p) \to THH(ku_p)$ factors through an isomorphism onto the Δ-fixed elements of $V(1)_* THH(ku_p)$,

$$i_* : V(1)_* THH(\ell_p) \xrightarrow{\cong} \big(V(1)_* THH(ku_p)\big)^{\Delta} \subset V(1)_* THH(ku_p), \tag{6.1}$$

see [1, 10.1]. The corresponding results hold also for the C_{p^n} or S^1 homotopy fixed points of THH, for the C_{p^n} or S^1 Tate construction on THH, and for TC and K, see [1, 10.2]. In the sequel, we identify $V(1)_* THH(\ell_p)$, $V(1)_* TC(\ell_p)$, etc. with their image under i_*. We have a similar statement for the various spectral sequences computing the $V(1)$-homotopy of these spectra.

Lemma 6.1 *Let $G = S^1$ or $G = C_{p^n}$, and let $E^*(G, \ell_p)$ and $E^*(G, ku_p)$ be the G homotopy fixed-point spectral sequences converging strongly to $V(1)_* THH(\ell_p)^{hG}$ and $V(1)_* THH(ku_p)^{hG}$, respectively. Then the morphism of spectral sequences induced by the map $\ell_p \to ku_p$ is equal to the inclusion of the Δ fixed points*

$$E^*(G, \ell_p) = \big(E^*(G, ku_p)\big)^{\Delta} \subset E^*(G, ku_p).$$

This holds also for the morphism induced on the G Tate spectral sequences converging to $V(1)_*THH(\ell_p)^{tG}$ and $V(1)_*THH(ku_p)^{tG}$, which is given by

$$\hat{E}^*(G,\ell_p) = \big(\hat{E}^*(G,ku_p)\big)^{\Delta} \subset \hat{E}^*(G,ku_p).$$

Proof The group Δ acts on ku_p by S-algebra maps, and it acts S^1-equivariently on $THH(ku_p)$. In particular Δ acts by morphisms of spectral sequences on $E^*(G,ku_p)$ and $\hat{E}^*(G,ku_p)$, and hence it suffices to prove that the claims hold at the level of the E^2-terms. This follows from (6.1). □

From now on, we will omit ku_p from the notation and just write $E^*(G)$ and $\hat{E}^*(G)$ for the G homotopy fixed-point and G Tate spectral sequences converging to $V(1)_*THH(ku_p)^{hG}$ and $V(1)_*THH(ku_p)^{tG}$, respectively.

At this point, we recall the notion of δ-weight introduced in [1, 8.2]. We fix a generator δ of the group Δ acting on ku_p, $K(ku_p)$, $THH(ku_p)$, $TC(ku_p)$, etc. The self-map δ_* of $V(1)_*ku_p = P_{p-1}(u)$ maps u to αu for some generator α of $\mathbb{F}_p^\times$. We say that a class $v \in V(1)_*K(ku_p)$ has δ-weight $i \in \mathbb{Z}/(p-1)$ if $\delta_*(v) = \alpha^i v$. The same convention holds for classes in $V(1)_*THH(ku_p)$, $V(1)_*TC(ku_p)$, etc. For example, the generators a_i and b_j of $V(1)_*THH(ku_p)$ given in (4.1) all have δ-weight 1, see [1, 10.1]. Similarly, it follows from its definition that $b \in V(1)_*K(ku_p)$ has δ-weight 1. Since δ_* is diagonalizable, we can reinterpret Lemma 6.1 by saying that each of these spectral sequences for ku_p has an extra $\mathbb{Z}/(p-1)$-grading given by the δ-weight, and that its homogeneous summand of δ-weight 0 consists of the corresponding spectral sequence for ℓ_p. Together with the internal and filtration degrees, the δ-weight endows the E^r-terms of these spectral sequences with a tri-grading that we will refer to in the computations below.

By a computation of McClure and Staffeldt [32], [2, 2.6], we have an isomorphism of $\mathbb{F}_p$-algebras

$$V(1)_*THH(\ell_p) \cong E(\lambda_1,\lambda_2) \otimes P(\mu).$$

The induced map $V(1)_*THH(\ell_p) \to V(1)_*THH(ku_p)$ sends λ_1 and μ to the classes with same name, and λ_2 to the class $a_1 b_1^{p-2}$.

Remark 6.2 In the sequel, we will frequently denote by λ_2 the class $a_1 b_1^{p-2}$.

The C_p-Tate spectral sequence

$$\hat{E}(C_p)^2_{s,t} = \hat{H}^{-s}\big(C_p, V(1)_t THH(ku_p)\big) \Rightarrow V(1)_{s+t} THH(ku_p)^{tC_p}$$

has an E_2-term given by

$$\hat{E}(C_p)^2 = P(t,t^{-1}) \otimes E(u_1) \otimes V(1)_*THH(ku_p)$$

with t in bidegree $(-2,0)$, u_1 in bidegree $(-1,0)$, and $w \in V(1)_t THH(ku_p)$ in bidegree $(0,t)$. Recall the description of $V(1)_* THH(ku_p)$ given in (4.1).

Lemma 6.3 *In the C_p Tate spectral sequence $\hat{E}^*(C_p)$ the classes λ_1, λ_2, b_1 and $t\mu$ are infinite cycles. There are non-zero differentials*

$$d^2(b_i) = (1-i)a_i t$$

$$d^{2p}(t^{1-p}) \doteq \lambda_1 \cdot t$$

$$d^{2p^2}(t^{p-p^2}) \doteq \lambda_2 \cdot t^p$$

$$d^{2p^2+1}(u_1 \cdot t^{-p^2}) \doteq t\mu$$

with $0 \le i \le p-1$. The spectral sequence collapses at the $\hat{E}^{2p^2+2}$-term, leaving

$$\hat{E}^\infty(C_p) = P(t^{\pm p^2}) \otimes E(\lambda_1, a_1) \otimes P_{p-1}(b_1)$$
$$\oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes \mathbb{F}_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 1\}.$$

Remark 6.4 Beware that in the lemma above, the index j appearing as a power of t runs over all integers, positive or negative, with specified p-adic valuation. The same remark holds for the Lemmas 6.10 and 6.12 below, and also for the power j of μ in Lemmas 6.11 and 6.13 below.

Proof We know from [2, Proposition 4.8] that $t\mu$ is an infinite cycle. The classes λ_1, λ_2 and b_1 are also infinite cycles, see the argument given at the top of [2, p. 21].

Let d be Connes' operator (4.3) on $V(1)_* THH(ku_p)$, and recall from above the notation $b_0 = u$. We have

$$d(b_0) = a_0,$$

and this relation is detected via the Hurewicz homomorphism in mod (p) homology, see [1, Sect. 9]. It follows from [37, Sect. 3.3] that in the S^1 homotopy fixed-point spectral sequence

$$E^2(S^1) = P(t) \otimes V(1)_* THH(ku_p) \Rightarrow V(1)_* THH(ku_p)^{hS^1}$$

we have a d^2-differential

$$d^2(b_0) = a_0 t.$$

Since $E^2(S^1)$ injects into $\hat{E}^2(C_p)$ via $R^h F$, this differential is also present in $\hat{E}^2(C_p)$. The differentials $d^2(b_i) = (1-i)a_i t$ for $i \neq 0$ follow easily from the case $i = 0$ and the multiplicative structure. Indeed $d^2(\mu) = 0$ for degree reasons, and hence $d^2(u^2\mu) = 2u\mu a_0 t$. From the relation $b_i b_{p-i} = u^2\mu$ we deduce that $d^2(b_i) = \alpha_i a_i t$ for some $\alpha_i \in \mathbb{F}_p$, because in $V(1)_* THH(ku_p)$ the equation $x b_{p-i} = u\mu a_0$ has $x = a_i$ as unique (homogeneous) solution. First, notice that $0 = d^2(b_1^{p-1}) = (p-1)\alpha_1\lambda_2$, so we have $\alpha_1 = 0$. Next, the relation $b_1 b_{p-1} = u^2\mu$ implies that $\alpha_{p-1} = 2$, while $b_1 b_i = u b_{i+1}$ for $i \le p-2$ implies that $\alpha_i = 1 + \alpha_{i+1}$. We deduce that $\alpha_i = 1 - i$, proving the claim on the d^2-differential, which leaves

$$\hat{E}^3(C_p) = P(t^{\pm 1}, t\mu) \otimes E(u_1, \lambda_1, a_1) \otimes P_{p-1}(b_1).$$

Lemma 6.1 determines the given next three non-zero differentials, by comparison with the case of the ℓ_p treated in [2, Sect. 5.5], and this takes care of the summand of δ-weight zero. The only algebra generators of $\hat{E}^3(C_p)$ of non-zero δ-weight are a_1 and b_1. We know that b_1 is an infinite cycle. In the S^1 Tate spectral sequence, using the known differentials, the tri-grading and the product, it is easy to see that a_1 survives to the E^{2p^2+2}-term. Therefore a_1 also survives to the E^{2p^2+2}-term in $\hat{E}^*(C_p)$, via the morphism of spectral sequences induced by F. The d^{2p} differential leaves

$$\hat{E}^{2p+1}(C_p) = P(t^{\pm p}, t\mu) \otimes E(u_1, \lambda_1, a_1) \otimes P_{p-1}(b_1),$$

and the d^{2p^2} differential leaves

$$\begin{aligned} \hat{E}^{2p^2+1}(C_p) = P(t^{\pm p^2}, t\mu) \otimes E(u_1, \lambda_1, a_1) \otimes P_{p-1}(b_1) \oplus E(u_1, \lambda_1) \\ \otimes P_{p-2}(b_1) \otimes P(t\mu) \otimes \mathbb{F}_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 1\}, \end{aligned}$$

as can be computed using the relation $a_1 \cdot b^{p-2} = \lambda_2$. Finally, d^{2p^2+1} leaves

$$\begin{aligned} \hat{E}^{2p^2+2}(C_p) = P(t^{\pm p^2}) \otimes E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \\ \oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes \mathbb{F}_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 1\}, \end{aligned}$$

and at this stage the spectral sequence collapses for bidegree reasons. □

Remark 6.5 The d^2-differential can also be determined by computing $d(b_i)$ for $i \ge 0$, using Connes' operator in Hochschild homology (c.f. [1, 3.4]).

Definition 6.6 We call a homomorphism of graded groups k-coconnected if it is an isomorphism in all dimensions greater than k and injective in dimension k.

Proposition 6.7 *The algebra map*

$$(\hat{\Gamma}_1)_* : V(1)_* THH(ku_p) \to V(1)_* THH(ku_p)^{tC_p}$$

factorizes as the localization away from μ, followed by an isomorphism

$$V(1)_* THH(ku_p)[\mu^{-1}] \to V(1)_* THH(ku_p)^{tC_p}$$

given by

$$\lambda_1 \mapsto \lambda_1, \ \mu \mapsto t^{-p^2}, \ b_i \mapsto t^{(1-i)p} b_1, \ \textit{and } a_i \mapsto t^{(1-i)p} a_1$$

for $0 \le i \le p-1$, *up to some non-zero scalar multiples. In particular the map* $(\hat{\Gamma}_1)_*$ *is* $(2p-2)$*-coconnected.*

Proof By naturality with respect to $\ell_p \to ku_p$ and by the computation of $(\hat{\Gamma}_1)_*$ for ℓ_p given in [2, Theorem 5.5], we know that the map $(\hat{\Gamma}_1)_*$ for ku_p satisfies

$$\lambda_1 \mapsto \lambda_1, \qquad \lambda_2 \mapsto \lambda_2 \quad \text{and} \quad \mu \mapsto t^{-p^2}.$$

In $V(1)_* THH(ku_p)$ we have multiplicative relations $u^{p-3} a_i b_j = \lambda_2$ for $i + j = p-1$, from which we deduce that $(\hat{\Gamma}_1)_*(u^k a_i) \neq 0$ and $(\hat{\Gamma}_1)_*(u^k b_i) \neq 0$ for any $0 \le k \le p-3$ and any $0 \le i \le p-1$. For degree reasons, this forces

$$(\hat{\Gamma}_1)_*(a_i) = t^{(1-i)p} a_1 \quad \text{and} \quad (\hat{\Gamma}_1)_*(b_i) = t^{(1-i)p} b_1$$

up to some non-zero scalar multiples. □

Corollary 6.8 *The canonical maps*

$$\Gamma_n : THH(ku_p)^{C_{p^n}} \to THH(ku_p)^{hC_{p^n}},$$

$$\hat{\Gamma}_n : THH(ku_p)^{C_{p^{n-1}}} \to THH(ku_p)^{tC_{p^n}},$$

$$\Gamma : TF(ku_p) \to THH(ku_p)^{hS^1},$$

$$\hat{\Gamma} : TF(ku_p) \to THH(ku_p)^{tS^1},$$

for $n \ge 1$ all induce $(2p-2)$*-coconnected maps in* $V(1)$*-homotopy.*

Proof The claims for Γ_n and $\hat{\Gamma}_n$ follow from Proposition 6.7 and the generalization of a theorem of Tsalidis [43] given in [13]. The claims for Γ and $\hat{\Gamma}$ follow by passage to homotopy limits. □

Definition 6.9 Let $r(n) = 0$ for all $n \le 0$, and let $r(n) = p^n + r(n-2)$ for all $n \ge 1$. Thus $r(2n-1) = p^{2n-1} + \cdots + p$ (odd powers) and $r(2n) = p^{2n} + \cdots + p^2$ (even powers).

Lemma 6.10 *In the C_{p^n} Tate spectral sequence $\hat{E}^*(C_{p^n})$ the classes λ_1, λ_2, b_1 and $t\mu$ are infinite cycles. There are non-zero differentials*

$$\begin{aligned} d^2(b_i) &= (1-i)a_i t \\ d^{2p}(t^{1-p}) &\doteq \lambda_1 \cdot t \\ d^{2p^2}(t^{p-p^2}) &\doteq \lambda_2 \cdot t^p \end{aligned}$$

with $0 \le i \le p-1$, leaving

$$\begin{aligned} \hat{E}^{2p^2+1}(C_{p^n}) = & P(t^{\pm p^2}) \otimes E(u_n, \lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu) \\ & \oplus E(u_n, \lambda_1) \otimes P_{p-2}(b_1) \otimes P(t\mu) \\ & \otimes \mathbb{F}_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 1\}. \end{aligned}$$

If $n \ge 2$, then for each $1 \le k \le n-1$ there is a triple of non-zero differentials

$$\begin{aligned} d^{2r(2k)+2}(b_1 t^j) &\doteq a_1 t^j \cdot t^{p^{2k}} \cdot (t\mu)^{r(2k-2)+1} \\ d^{2r(2k+1)}(t^{p^{2k}-p^{2k+1}}) &\doteq \lambda_1 \cdot t^{p^{2k}} \cdot (t\mu)^{r(2k-1)} \\ d^{2r(2k+2)}(t^{p^{2k+1}-p^{2k+2}}) &\doteq \lambda_2 \cdot t^{p^{2k+1}} \cdot (t\mu)^{r(2k)} \end{aligned}$$

with $v_p(j) = 2k-1$, leaving

$$\begin{aligned} \hat{E}^{2r(2k+2)+1}(C_{p^n}) = & P(t^{\pm p^{2k+2}}) \otimes E(u_n, \lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu) \\ & \oplus E(u_n, \lambda_1) \otimes P_{p-2}(b_1) \otimes P(t\mu) \\ & \otimes \mathbb{F}_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 2k+1\} \\ & \oplus \bigoplus_{1 \le m \le k} \hat{T}_m(C_{p^n}), \end{aligned}$$

where

$$\begin{aligned}\hat{T}_m(C_{p^n}) = & E(u_n, \lambda_1) \otimes P_{r(2m)}(t\mu) \otimes \mathbb{F}_p\{\lambda_2 t^j \mid v_p(j) = 2m+1\} \\ & \oplus E(u_n, a_1) \otimes P_{p-1}(b_1) \otimes P_{r(2m-1)}(t\mu) \\ & \otimes \mathbb{F}_p\{\lambda_1 t^j \mid v_p(j) = 2m\} \\ & \oplus E(u_n, \lambda_1) \otimes P_{p-2}(b_1) \otimes P_{r(2m-2)+1}(t\mu) \\ & \otimes \mathbb{F}_p\{a_1 t^j \mid v_p(j) = 2m-1\}.\end{aligned}$$

For $n \geq 1$, *there is a last non-zero differential*

$$d^{2r(2n)+1}(u_n \cdot t^{-p^{2n}}) \doteq (t\mu)^{r(2n-2)+1}$$

after which the spectral sequence collapses, *leaving*

$$\begin{aligned}\hat{E}^\infty(C_{p^n}) = & P(t^{\pm p^{2n}}) \otimes E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P_{r(2n-2)+1}(t\mu) \\ & \oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes P_{r(2n-2)+1}(t\mu) \\ & \otimes \mathbb{F}_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 2n-1\} \\ & \oplus \bigoplus_{1 \leq m \leq n-1} \hat{T}_m(C_{p^n}).\end{aligned}$$

Next, we describe the C_{p^n} homotopy fixed-point spectral sequence $E^*(C_{p^n})$ for $THH(ku_p)$. It is algebraically easier to describe the E^r-terms of the C_{p^n} homotopy fixed-point spectral sequence for $THH(ku_p)^{tC_p}$, which we denote abusively by

$$\mu^{-1}E^*(C_{p^n}) \Rightarrow V(1)_*(THH(ku_p)^{tC_p})^{hC_{p^n}},$$

compare with [2, p. 23]. We know from Proposition 6.7 that the map

$$\hat{\Gamma}_1^{hC_{p^n}} : THH(ku_p)^{hC_{p^n}} \to (THH(ku_p)^{tC_p})^{hC_{p^n}}$$

induces a morphism of spectral sequences

$$E^*(C_{p^n}) \to \mu^{-1}E^*(C_{p^n})$$

which on E^2-terms (but not on higher terms) indeed corresponds to inverting μ. By the same Proposition and by strong convergence of the spectral sequences, the map $\hat{\Gamma}_1^{hC_{p^n}}$ induces a $(2p-2)$-coconnected homomorphism in $V(1)$-homotopy.

Lemma 6.11 *In the C_{p^n} homotopy fixed-point spectral sequence $\mu^{-1}E^*(C_{p^n})$ the classes λ_1, λ_2, b_1 and $t\mu$ are infinite cycles. There are non-zero differentials*

$$d^2(b_i) = (1-i)a_i t$$
$$d^{2p}(\mu^{p-1}) \doteq \lambda_1 \cdot \mu^{-1} \cdot (t\mu)^p$$
$$d^{2p^2}(\mu^{p^2-p}) \doteq \lambda_2 \cdot \mu^{-p} \cdot (t \setminus \mu)^{p^2}$$

with $0 \le i \le p-1$, leaving

$$\begin{aligned}\mu^{-1}E^{2p^2+1}(C_{p^n}) = P(\mu^{\pm p^2}) \otimes E(u_n, \lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu)\\ \oplus E(u_n, \lambda_1) \otimes P_{p-2}(b_1) \otimes P(t\mu)\\ \otimes \mathbb{F}_p\{a_1\mu^j, b_1\mu^j \mid v_p(j) = 1\} \oplus T_1(C_{p^n}),\end{aligned}$$

where

$$\begin{aligned}T_1(C_{p^n}) = E(u_n, \lambda_1) \otimes P_{p^2}(t\mu) \otimes \mathbb{F}_p\{\lambda_2\mu^j \mid v_p(j) = 1\}\\ \oplus E(u_n, a_1) \otimes P_{p-1}(b_1) \otimes P_p(t\mu) \otimes \mathbb{F}_p\{\lambda_1\mu^j \mid v_p(j) = 0\}\\ \oplus E(u_n, \lambda_1) \otimes P_{p-2}(b_1) \otimes P(\mu^{\pm 1})\\ \otimes \mathbb{F}_p\{a_i \mid 0 \le i \le p-1,\ i \ne 1\}.\end{aligned}$$

If $n \ge 2$, then for each $2 \le k \le n$ there is a triple of non-zero differentials

$$d^{2r(2k-2)+2}(b_1\mu^j) \doteq a_1\mu^j \cdot \mu^{-p^{2k-2}} \cdot (t\mu)^{r(2k-2)+1}$$
$$d^{2r(2k-1)}(\mu^{p^{2k-1}-p^{2k-2}}) \doteq \lambda_1 \cdot \mu^{-p^{2k-2}} \cdot (t\mu)^{r(2k-1)}$$
$$d^{2r(2k)}(\mu^{p^{2k}-p^{2k-1}}) \doteq \lambda_2 \cdot \mu^{-p^{2k-1}} \cdot (t\mu)^{r(2k)}$$

with $v_p(j) = 2k-3$, leaving

$$\begin{aligned}\mu^{-1}E^{2r(2k)+1}(C_{p^n}) = P(\mu^{\pm p^{2k}}) \otimes E(u_n, \lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu)\\ \oplus E(u_n, \lambda_1) \otimes P_{p-2}(b_1) \otimes P(t\mu)\\ \otimes \mathbb{F}_p\{a_1\mu^j, b_1\mu^j \mid v_p(j) = 2k-1\}\\ \oplus \bigoplus_{1 \le m \le k} T_m(C_{p^n}),\end{aligned}$$

where for $m \geq 2$ we have

$$\begin{aligned} T_m(C_{p^n}) = E(u_n, \lambda_1) \otimes P_{r(2m)}(t\mu) \otimes \mathbb{F}_p\{\lambda_2\mu^j \mid v_p(j) = 2m-1\} \\ \oplus E(u_n, a_1) \otimes P_{p-1}(b_1) \otimes P_{r(2m-1)}(t\mu) \\ \otimes \mathbb{F}_p\{\lambda_1\mu^j \mid v_p(j) = 2m-2\} \\ \oplus E(u_n, \lambda_1) \otimes P_{p-2}(b_1) \otimes P_{r(2m-2)+1}(t\mu) \\ \otimes \mathbb{F}_p\{a_1\mu^j \mid v_p(j) = 2m-3\}. \end{aligned}$$

For $n \geq 1$, there is a last non-zero differential

$$d^{2r(2n)+1}(u_n \cdot \mu^{p^{2n}}) \doteq (t\mu)^{r(2n)+1}$$

after which the spectral sequence collapses, leaving

$$\begin{aligned} \mu^{-1}E^\infty(C_{p^n}) = P(\mu^{\pm p^{2n}}) \otimes E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P_{r(2n)+1}(t\mu) \\ \oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes P_{r(2n)+1}(t\mu) \\ \otimes \mathbb{F}_p\{a_1\mu^j, b_1\mu^j \mid v_p(j) = 2n-1\} \\ \oplus \bigoplus_{1 \leq m \leq n} T_m(C_{p^n}). \end{aligned}$$

Proof We prove these two lemmas by induction on n, showing that Lemma 6.10 for C_{p^n} implies Lemma 6.11 for C_{p^n}, which in turn implies Lemma 6.10 for $C_{p^{n+1}}$. The induction starts with Lemma 6.10 for C_p, which is the content of Lemma 6.3. Let us therefore assume given $n \geq 1$ such that Lemma 6.10 holds for C_{p^n}. The homotopy restriction map

$$R^h : THH(ku_p)^{hC_{p^n}} \to THH(ku_p)^{tC_{p^n}}$$

induces a morphism of spectral sequences $(R^h)^* : E^*(C_{p^n}) \to \hat{E}^*(C_{p^n})$, which at the E^2-terms corresponds to inverting the class $t \in E^2_{-2,0}(C_{p^n})$,

$$(R^h)^2 : E^2(C_{p^n}) \subset E^2(C_{p^n})[t^{-1}] \cong \hat{E}^2(C_{p^n}), \tag{6.2}$$

and can be pictured as the inclusion of the second quadrant into the upper-half plane. As we will see below, although $(R^h)^r$ is not injective for $r \geq 3$, it detects all the non-trivial differentials of $E^r(C_{p^n})$. Taking into account the multiplicative structure and the fact that $\lambda_1, \lambda_2, b_1$ and $t\mu$ are infinite cycles, we claim that these differential are given by

$$d^2(b_i) = (1-i)a_i t$$
$$d^{2p}(t) \doteq \lambda_1 \cdot t^{1+p}$$
$$d^{2p^2}(t^p) \doteq \lambda_2 \cdot t^{p+p^2}$$

with $0 \le i \le p-1$,

$$d^{2r(2k)+2}(b_1 t^j) \doteq a_1 t^j \cdot t^{p^{2k}} \cdot (t\mu)^{r(2k-2)+1}$$
$$d^{2r(2k+1)}(t^{p^{2k}}) \doteq \lambda_1 \cdot t^{p^{2k}+p^{2k+1}} \cdot (t\mu)^{r(2k-1)}$$
$$d^{2r(2k+2)}(t^{p^{2k+1}}) \doteq \lambda_2 \cdot t^{p^{2k+1}+p^{2k+2}} \cdot (t\mu)^{r(2k)}$$

if $n \ge 2$, $1 \le k \le n-1$ and $v_p(j) = 2k-1$ with $i \ge 0$, and finally

$$d^{2r(2n)+1}(u_n) \doteq (t\mu)^{r(2n-2)+1} \cdot t^{p^{2n}}.$$

To prove this claim, we assume that some $r \ge 2$ is given, and that $E^r(C_{p^n})$ has been computed using the differentials $d^{r'}$ above with $r' < r$. The class $t\mu$ is an infinite cycle, and $E^r(C_{p^n})$ is a $P(t\mu)$-module. Our choice of generators induces a decomposition $E^r(C_{p^n}) \cong F^r(C_{p^n}) \oplus T^r(C_{p^n})$, where $F^r(C_{p^n})$ is a free $P(t\mu)$-module and $T^r(C_{p^n})$ is a $t\mu$-torsion module. By inspection, the non-zero elements of $T^r(C_{p^n})$ are concentrated in filtration degrees s with $-r < s \le 0$, so they cannot be boundaries. They cannot support non-zero differentials either since a $t\mu$-torsion class cannot map to a non-torsion class. Thus the differential d^r maps $F^r(C_{p^n})$ to itself and $T^r(C_{p^n})$ to zero. The morphism $(R^h)^r$ maps $F^r(C_{p^n})$ injectively into $\hat{E}^r(C_{p^n})$, and it therefore detects the non-zero differentials of $E^r(C_{p^n})$ as the non-zero differential of $\hat{E}^r(C_{p^n})$ which lie in the second quadrant. These are precisely the differentials given above. By induction on r, this determines all the non-trivial differentials of $E^*(C_{p^n})$. In the μ-inverted homotopy fixed-point spectral sequence $\mu^{-1}E^*(C_{p^n})$, these can be rewritten as the claimed differentials. This proves Lemma 6.11 for C_{p^n}.

We now turn to the proof of Lemma 6.10 for $C_{p^{n+1}}$. In the Tate spectral sequence $\hat{E}^*(C_{p^n})$ the first non-zero differential of odd length originating from a column of odd s-filtration is $d^{2r(2n)+1}$. By [2, Lemma 5.2] the spectral sequences $\hat{E}^*(C_{p^n})$ and $\hat{E}^*(C_{p^{n+1}})$ are abstractly isomorphic up to the $E^{2r(2n)+1}$-term included. The Frobenius map

$$F : THH(ku_p)^{tC_{p^{n+1}}} \to THH(ku_p)^{tC_{p^n}}$$

induces a morphism of the corresponding Tate spectral sequences, which on E^r-terms with $2 \le r \le 2r(2n)+1$ maps the columns of even s-filtration isomorphically. This detects all the claimed differentials of $\hat{E}^r(C_{p^{n+1}})$ for $2 \le r \le 2r(2n)$, and leaves

$$\hat{E}^{2r(2n)+1}(C_{p^{n+1}}) = \hat{F}^{2r(2n)+1}(C_{p^{n+1}}) \oplus \bigoplus_{m=1}^{n-1} \hat{T}_m(C_{p^{n+1}}),$$

where $\hat{F}^{2r(2n)+1}(C_{p^{n+1}})$ is the $t\mu$-torsion free summand

$$\begin{aligned}\hat{F}^{2r(2n)+1}(C_{p^{n+1}}) = P(t^{\pm p^{2n}}) &\otimes E(u_{n+1}, \lambda_1) \otimes P(t\mu) \\ &\otimes \Big(P_{p-1}(b_1) \otimes E(a_1) \oplus P_{p-2}(b_1) \\ &\otimes \mathbb{F}_p\{a_1 t^{-ip^{2n-1}}, b_1 t^{-ip^{2n-1}} \,|\, 0 < i < p\}\Big).\end{aligned}$$

The non-zero $t\mu$-torsion elements of $\hat{E}^{2r(2n)+1}(C_{p^{n+1}})$ are concentrated in internal degrees t with $0 \le t < 2r(2n)$. In particular these elements cannot be boundaries, and they cannot map to non-$t\mu$-torsion elements. As in the case of the homotopy fixed-point spectral sequence above, we deduce that for $r \ge 2r(2n)+1$ the differential d^r can only affect the summand $\hat{F}^{2r(2n)+1}(C_{p^{n+1}})$. By Lemma 6.1 the summand of δ-weight 0 of $\hat{E}^*(C_{p^{n+1}})$ is equal to the image of the injective morphism of spectral sequences

$$\hat{E}^*(C_{p^{n+1}}, \ell_p) \to \hat{E}^*(C_{p^{n+1}}, ku_p) = \hat{E}^*(C_{p^{n+1}})$$

induced by the map $\ell_p \to ku_p$. Therefore, by [2, Theorem 6.1], the differentials affecting the summand of δ-weight 0 of $\hat{F}^{2r(2n)+1}(C_{p^{n+1}})$ at a later stage are given by

$$\begin{aligned} d^{2r(2n+1)}(t^{p^{2n}-p^{2n+1}}) &\doteq \lambda_1 \cdot t^{p^{2n}} \cdot (t\mu)^{r(2n-1)} \\ d^{2r(2n+2)}(t^{p^{2n+1}-p^{2n+2}}) &\doteq \lambda_2 \cdot t^{p^{2n+1}} \cdot (t\mu)^{r(2n)} \\ d^{2r(2n+2)+1}(u_{n+1} \cdot t^{-p^{2n+2}}) &\doteq (t\mu)^{r(2n)+1}, \end{aligned} \tag{6.3}$$

together with the multiplicative structure and the fact that $t\mu$ is an infinite cycle. It remains to prove that from the $E^{2r(2n)+1}$-term on, the only non-zero differentials supported by homogeneous algebra generators of δ-weight 1 are given by

$$d^{2r(2n)+2}(b_1 t^j) \doteq a_1 t^j \cdot t^{p^{2n}} \cdot (t\mu)^{r(2n-2)+1} \tag{6.4}$$

for $v_p(j) = 2n - 1$. First, notice that for tri-degree reasons $d^{2r(2n)+1} = 0$, so that $\hat{F}^{2r(2n)+2}(C_{p^{n+1}}) = \hat{F}^{2r(2n)+1}(C_{p^{n+1}})$. To detect the differential (6.4) we make use of the $(2p-2)$-coconnected map

$$(\hat{\Gamma}_{n+1})_* : V(1)_*THH(ku_p)^{C_{p^n}} \to V(1)_*THH(ku_p)^{tC_{p^{n+1}}},$$

and argue as in [2, proof of 6.1]. There is a commutative diagram

$$\begin{tikzcd}
THH(ku_p)^{hC_{p^n}} \arrow[d, "F^n"] & THH(ku_p)^{C_{p^n}} \arrow[l, "\Gamma_n"'] \arrow[r, "\hat{\Gamma}_{n+1}"] \arrow[d, "F^n"] & THH(ku_p)^{tC_{p^{n+1}}} \arrow[d, "F^n"] \\
THH(ku_p) & THH(ku_p) \arrow[l, "\Gamma_0"', "="] \arrow[r, "\hat{\Gamma}_1"] & THH(ku_p)^{tC_p}
\end{tikzcd}$$

where the vertical arrows are the n-fold Frobenius maps. The left-hand Frobenius is given in $V(1)$-homotopy on the associated graded by the edge homomorphism

$$E^\infty_{*,*}(C_{p^n}) \to E^\infty_{0,*}(C_{p^n}) \subset E^2_{0,*}(C_{p^n}) = V(1)_*THH(ku_p),$$

which is known by induction hypothesis. For each $0 < \ell < p$ there is a direct summand

$$P_{r(2n-2)+1}(t\mu)\{a_1\mu^{\ell p^{2n-3}}\} \subset E^\infty_{*,*}(C_{p^n}),$$

and $a_1\mu^{\ell p^{2n-3}}$ maps by F^n_* to the class with same name in $V(1)_*THH(ku_p)$. Since $(\Gamma_n)_*$ is $(2p-2)$-coconnected, there is a class $x_\ell \in V(1)_*THH(ku_p)^{C_{p^n}}$ with $F^n_*(x_\ell) = a_1\mu^{\ell p^{2n-3}}$ in $V(1)_*THH(ku_p)$. In $E^\infty(C_{p^n})$ we have no non-zero class of same total degree, same δ-weight and lower s-filtration than

$$(t\mu)^{r(2n-2)+1} \cdot a_1\mu^{\ell p^{2n-3}},$$

which forces $v_2^{r(2n-2)+1}x_\ell = 0$ in $V(1)_*THH(ku_p)^{C_{p^n}}$. By Proposition 6.7, the class $(\hat{\Gamma}_1 F^n)_*(x_\ell)$ is represented by $a_1 t^{-\ell p^{2n-1}} \in \hat{E}^\infty(C_p)$, and therefore $(\hat{\Gamma}_{n+1})_*(x_\ell)$ must be detected in s-filtration $2\ell p^{2n-1}$ or higher. The only suitable class in $\hat{E}^{2r(2n)+2}(C_{p^{n+1}})$ is $a_1 t^{-\ell p^{2n-1}}$, which therefore is a permanent cycle representing $(\hat{\Gamma}_{n+1})_*(x_\ell)$. Notice for later use that the same argument shows that

$$a_1 \in \hat{E}^{2r(2n)+2}_{0,2p+3}(C_{p^{n+1}})$$

is a permanent cycle. The map $(\hat{\Gamma}_{n+1})_*$ is an isomorphism in degrees larger than $2p-2$, and the relation $v_2^{r(2n-2)+1}(\hat{\Gamma}_{n+1})_*(x_\ell) = 0$ implies that the infinite cycle $(t\mu)^{r(2n-2)+1} \cdot a_1 t^{-\ell p^{2n-1}}$, of total degree $2p^{2n} + 2\ell p^{2n-1} + 2p + 1$ and of δ-weight 1, is a boundary. On the other hand, the component of $\hat{E}^{2r(2n)+1}(C_{p^{n+1}})$ of total degree $2p^{2n} + 2\ell p^{2n-1} + 2p + 2$, of δ-weight 1 and of s-filtration degree exceeding by at least $2r(2n) + 2$ the s-filtration degree of $(t\mu)^{r(2n-2)+1} \cdot a_1 t^{-\ell p^{2n-1}}$ reduces to

$$\mathbb{F}_p\{b_1 t^{-\ell p^{2n-1}} \cdot t^{-p^{2n}}\}.$$

This proves the existence of a non-zero differential

$$d^{2r(2n)+2}(b_1 t^{-\ell p^{2n-1}} \cdot t^{-p^{2n}}) \doteq (t\mu)^{r(2n-2)+1} \cdot a_1 t^{-\ell p^{2n-1}}$$

for $0 < \ell < p$. Since $t^{p^{2n}}$ is a unit and a cycle we obtain the claimed differentials (6.4). This leaves

$$\begin{aligned} \hat{E}^{2r(2n)+3}(C_{p^{n+1}}) = \hat{F}^{2r(2n)+3}(C_{p^{n+1}}) \\ &\oplus E(u_{n+1}, \lambda_1) \otimes P_{p-2}(b_1) \otimes P_{r(2n-2)+1}(t\mu) \\ &\otimes \mathbb{F}_p\{a_1 t^j \mid v_p(j) = 2n-1\} \\ &\oplus \bigoplus_{m=1}^{n-1} \hat{T}_m(C_{p^{n+1}}), \end{aligned}$$

with a $t\mu$-torsion free summand

$$F^{2r(2n)+3}(C_{p^{n+1}}) = P(t^{\pm p^{2n}}) \otimes E(u_{n+1}, \lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu).$$

Again, further differentials can only affect the summand $F^{2r(2n)+3}(C_{p^{n+1}})$. Since b_1 and a_1 are infinite cycles, the next non-zero differentials are $d^{2r(2n+1)}$ and $d^{2r(2n+2)}$, as given in (6.3), leaving

$$\hat{E}^{2r(2n+2)+1}(C_{p^{n+1}}) = \hat{F}^{2r(2n+2)+1}(C_{p^{n+1}}) \oplus \bigoplus_{m=1}^{n} \hat{T}_m(C_{p^{n+1}}),$$

with

$$\begin{aligned} \hat{F}^{2r(2n+2)+1}(C_{p^{n+1}}) = P(t^{\pm p^{2n+2}}) \otimes E(u_{n+1}, \lambda_1) \otimes P(t\mu) \\ &\otimes \Big(P_{p-1}(b_1) \otimes E(a_1) \oplus P_{p-2}(b_1) \\ &\otimes \mathbb{F}_p\{a_1 t^{-ip^{2n+1}}, b_1 t^{-ip^{2n+1}} \mid 0 < i < p\}\Big). \end{aligned}$$

Notice that for tri-degree reasons, the classes $a_1 t^{-ip^{2n+1}}$ and $b_1 t^{-ip^{2n+1}}$ are cycles at the $E^{2r(2n+2)+1}$-stage. The third differential of (6.3) remains, after which the spectral sequence collapses for bidegree reasons, leaving

$$\begin{aligned}\hat{E}^\infty(C_{p^{n+1}}) = P(t^{\pm p^{2(n+1)}}) &\otimes E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P_{r(2n)+1}(t\mu)\\ &\oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes P_{r(2n)+1}(t\mu)\\ &\otimes \mathbb{F}_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 2n+1\}\\ &\oplus \bigoplus_{1\le m\le n} \hat{T}_m(C_{p^{n+1}}),\end{aligned}$$

as claimed. This completes the induction step and the proof of Lemmas 6.10 and 6.11. □

Taking the limit over the Frobenius maps we obtain the following two lemmas.

Lemma 6.12 *The associated graded $\hat{E}^\infty(S^1)$ of $V(1)_* THH(ku_p)^{tS^1}$ is given by*

$$\hat{E}^\infty(S^1) = E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu) \oplus \bigoplus_{m\ge 1} \hat{T}_m(S^1),$$

where

$$\begin{aligned}\hat{T}_m(S^1) &= E(\lambda_1) \otimes P_{r(2m)}(t\mu) \otimes \mathbb{F}_p\{\lambda_2 t^j \mid v_p(j) = 2m+1\}\\ &\oplus E(a_1) \otimes P_{p-1}(b_1) \otimes P_{r(2m-1)}(t\mu) \otimes \mathbb{F}_p\{\lambda_1 t^j \mid v_p(j) = 2m\}\\ &\oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes P_{r(2m-2)+1}(t\mu)\\ &\otimes \mathbb{F}_p\{a_1 t^j \mid v_p(j) = 2m-1\}.\end{aligned}$$

Lemma 6.13 *The associated graded $E^\infty(S^1)$ of $V(1)_* THH(ku_p)^{hS^1}$ is mapped by a $(2p-2)$-coconnected homomorphism to*

$$\mu^{-1} E^\infty(S^1) = E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu) \oplus \bigoplus_{m\ge 1} T_m(S^1),$$

where

$$\begin{aligned}T_1(S^1) = &E(\lambda_1) \otimes P_{p^2}(t\mu) \otimes \mathbb{F}_p\{\lambda_2 \mu^j \mid v_p(j) = 1\}\\ &\oplus E(a_1) \otimes P_{p-1}(b_1) \otimes P_p(t\mu)\\ &\otimes \mathbb{F}_p\{\lambda_1 \mu^j \mid v_p(j) = 0\}\\ &\oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes P(\mu^{\pm 1})\\ &\otimes \mathbb{F}_p\{a_i \mid 0 \le i \le p-1,\ i \ne 1\}\end{aligned}$$

and, for $m \geq 2$,

$$\begin{aligned} T_m(S^1) = & E(\lambda_1) \otimes P_{r(2m)}(t\mu) \otimes \mathbb{F}_p\{\lambda_2\mu^j \mid v_p(j) = 2m-1\} \\ & \oplus E(a_1) \otimes P_{p-1}(b_1) \otimes P_{r(2m-1)}(t\mu) \\ & \otimes \mathbb{F}_p\{\lambda_1\mu^j \mid v_p(j) = 2m-2\} \\ & \oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes P_{r(2m-2)+1}(t\mu) \\ & \otimes \mathbb{F}_p\{a_1\mu^j \mid v_p(j) = 2m-3\}. \end{aligned}$$

7 Topological cyclic homology

We now evaluate the restriction map $R : TF(ku_p) \to TF(ku_p)$ in $V(1)$-homotopy. Consider the homotopy commutative diagram

$$\begin{tikzcd} TF(ku_p) \arrow[r, "R"] \arrow[d, "\Gamma"] & TF(ku_p) \arrow[r, "\Gamma"] \arrow[d, "\hat{\Gamma}"] & THH(ku_p)^{hS^1} \arrow[d, "(\hat{\Gamma}_1)^{hS^1}"] \\ THH(ku_p)^{hS^1} \arrow[r, "R^h"] & THH(ku_p)^{tS^1} \arrow[r, "G"] & (THH(ku_p)^{tC_p})^{hS^1} \end{tikzcd}$$

displayed in [2, p. 27], and with G a $V(1)$-equivalence. By the argument in [2, Lemma 7.5], we know that on $V(1)_*TF(ku_p)$ the profinite topology coincides with the topology induced via Γ_* by the spectral sequence filtration of $V(1)_*THH(ku_p)^{hS^1}$, and that the restriction map

$$R_* : V(1)_*TF(ku_p) \to V(1)_*TF(ku_p)$$

is continuous in degrees larger than $2p-2$. In this range of degrees, we identify $V(1)_*TF(ku_p)$ with $V(1)_*THH(ku_p)^{hS^1}$ via the homeomorphism Γ_*. Under this identification R_* corresponds to $(\Gamma_*\hat{\Gamma}_*^{-1})R_*^h$, and we first describe R_*^h and $\Gamma_*\hat{\Gamma}_*^{-1}$ separately.

Lemma 7.1 *In total degrees larger than* $2p-2$, *the morphism*

$$(R^h)^\infty : E^\infty(S^1) \to \hat{E}^\infty(S^1)$$

has the following properties.

(a) *It maps* $E(\lambda_1, a_1) \otimes P_{p-1}(b) \otimes P(t\mu)$ *isomorphically to the summand with same name*;

(b) *It maps* $E(\lambda_1) \otimes P_{r(k)}(t\mu) \otimes \mathbb{F}_p\{\lambda_2 \mu^{-dp^{k-1}}\}$ *onto*

$$E(\lambda_1) \otimes P_{r(k-2)}(t\mu) \otimes \mathbb{F}_p\{\lambda_2 t^{dp^{k-1}}\}$$

and $E(\lambda_1) \otimes P_{p-2}(b) \otimes P_{r(k)+1}(t\mu) \otimes \mathbb{F}_p\{a_1 \mu^{-dp^{k-1}}\}$ *onto*

$$E(\lambda_1) \otimes P_{p-2}(b) \otimes P_{r(k-2)+1}(t\mu) \otimes \mathbb{F}_p\{a_1 t^{dp^{k-1}}\}$$

for $k \geq 2$ *even and* $0 < d < p$;

(c) *It maps* $E(a_1) \otimes P_{p-1}(b) \otimes P_{r(k)}(t\mu) \otimes \mathbb{F}_p\{\lambda_1 \mu^{-dp^{k-1}}\}$ *onto*

$$E(a_1) \otimes P_{p-1}(b) \otimes P_{r(k-2)}(t\mu) \otimes \mathbb{F}_p\{\lambda_1 t^{dp^{k-1}}\}$$

for $k \geq 3$ *odd and* $0 < d < p$;

(d) *It maps the remaining summands to zero.*

Proof This follows from the description of $(R^h)^2$, see (6.2). □

Lemma 7.2 *In degrees larger then* $2p-2$, *the homomorphism* $\Gamma_* \hat{\Gamma}_*^{-1}$ *maps*

(a) *the classes in* $V(1)_* THH(ku_p)^{tS^1}$ *represented in* $\hat{E}^\infty(S^1)$ *by*

$$\lambda_1^{\epsilon_1} a_1^{\epsilon_2} b^k (t\mu)^m t^i$$

for $v_p(i) \neq 1$, ϵ_1 *and* $\epsilon_2 \in \{0, 1\}$, $0 \leq k \leq p-2$ *and* $m \geq 0$, *to classes in* $V(1)_* THH(ku_p)^{hS^1}$ *represented in* $E^\infty(S^1)$ *by*

$$\lambda_1^{\epsilon_1} a_1^{\epsilon_2} b^k (t\mu)^m \mu^j$$

with $i + p^2 j = 0$, *up to multiplication with a unit in* $\mathbb{F}_p$;

(b) *the classes in* $V(1)_* THH(ku_p)^{tS^1}$ *represented in* $\hat{E}^\infty(S^1)$ *by*

$$\lambda_1^{\epsilon_1} b^k a_1 t^i$$

for $v_p(i) = 1$, $\epsilon_1 \in \{0, 1\}$ *and* $0 \leq k \leq p-3$, *to classes in* $V(1)_* THH(ku_p)^{hS^1}$ *represented in* $E^\infty(S^1)$ *by*

$$\lambda_1^{\epsilon_1} b^k \mu^l a_j$$

with $i = (1-j)p - lp^2$ *for* $0 \leq j \leq p-1$ *such that* $j \neq 1$, *up to multiplication with a unit in* $\mathbb{F}_p$.

Proof The proof is similar to the proof of [2, Proposition 7.4], and we omit it. □

Definition 7.3 We recall from [2, Theorem 9.1] that there are classes $\lambda_1 t^{p-1}$, λ_1 and λ_2 in $V(1)_*K(\ell_p) \subset V(1)_*K(ku_p)$, of degree 1, $2p-1$ and $2p^2-1$, respectively. We denote by

$$\widetilde{\lambda_1 t^{p-1}}, \qquad \tilde{\lambda}_1 \quad \text{and} \quad \tilde{\lambda}_2$$

their image in $V(1)_*TF(ku_p)$ under tr_{F*}. The latter classes are represented by

$$\lambda_1 t^{p-1} = (t\mu)^{p-1} \cdot \lambda_1 \mu^{1-p}, \qquad \lambda_1 \quad \text{and} \quad \lambda_2$$

in $E^\infty(S^1)$, respectively, see [2, Theorem 8.4]. We further denote by b and v_2 the image in $V(1)_*TF(ku_p)$ under tr_{F*} of the classes with same name in $V(1)_*K(ku_p)$. These classes are represented by b_1 and $t\mu$ in $E^\infty(S^1)$, respectively, see Lemma 4.4 and [2, Proposition 4.8].

Lemma 7.4 *There exists a unique class $\tilde{a}_1 \in V(1)_{2p+3}TF(ku_p)$ with the following two properties*:

(a) $\tilde{a}_1$ *has δ-weight* 1 *and* $b^{p-2}\tilde{a}_1 = \tilde{\lambda}_2$,
(b) $R_*(\tilde{a}_1) = \tilde{a}_1$.

Moreover, this class $\tilde{a}_1$ is represented by a_1 in $E^\infty(S^1)$.

Proof For $i = 0$ or 1, let us denote by $T_*^{(i)}$ and $\ker(R-1)_*^{(i)}$ the summand of δ-weight i of $V(1)_*TF(ku_p)$ and $\ker(R-1)_* \subset V(1)_*TF(ku_p)$, respectively. We make the following claims:

(1) The homomorphism given by multiplication with b^{p-2} on $T_{2p+3}^{(1)}$ fits in a short exact sequence

$$0 \to \mathbb{F}_p\{z\} \to T_{2p+3}^{(1)} \xrightarrow{b^{p-2}} T_{2p^2-1}^{(0)} \to 0,$$

where the class z is represented by $b_1 \cdot (t\mu)^{p-1} \cdot \lambda_1 \mu^{1-p}$ in $E^\infty(S^1)$;
(2) The class z does not belong to $\ker(R-1)_*$.

Using these claims, it is easy to deduce that multiplication with b^{p-2} restricts to an isomorphism

$$\ker(R-1)_{2p+3}^{(1)} \xrightarrow{\cong} \ker(R-1)_{2p^2-1}^{(0)}.$$

We have $\tilde{\lambda}_2 \in \ker(R-1)_{2p^2-1}^{(0)}$ since $\tilde{\lambda}_2$ has δ-weight 0 and is in the image of tr_{F*}. Therefore, there is a unique pre-image $\tilde{a}_1 \in \ker(R-1)_{2p+3}^{(1)}$ of $\tilde{\lambda}_2 \in \ker(R-1)_{2p^2-1}^{(0)}$, or, in other words, there is a unique class $\tilde{a}_1 \in$

$V(1)_{2p+3}TF(ku_p)$ with properties (a) and (b). Moreover, $\tilde{\lambda}_2$ is represented in $E^\infty(S^1)$ in filtration zero by $\lambda_2 = b_1^{p-2}a_1$, and we deduce that $\tilde{a}_1$ must be represented in filtration zero by a_1. Thus this lemma follows from claims (1) and (2), which we now prove.

First, notice that the group $T_*^{(i)}$ inherits via Γ_* the spectral sequence filtration of $V(1)_*THH(ku_p)^{hS^1}$. Denoting by $E^\infty(S^1)_*^{(i)}$ its associated graded, we know from Lemma 6.13 that

$$E^\infty(S^1)_{2p+3}^{(1)} = \mathbb{F}_p\{a_1, b_1 \cdot x_n \,|\, n \geq 0\} \quad \text{and}$$
$$E^\infty(S^1)_{2p^2-1}^{(0)} = \mathbb{F}_p\{\lambda_2, t\mu \cdot x_n \,|\, n \geq 1\},$$

where $x_n = (t\mu)^{r(2n+1)-r(2n)-1} \cdot \lambda_1 \mu^{(1-p)p^{2n}}$.

Next, the relation $b'^p + v_2 b' = 0$ in $V(1)_*K(\mathbb{Z}, 3)$, established in Proposition 2.7, maps under $\mathrm{tr}_{F*}\phi_*$ to the relation $b^p + v_2 b = 0$ in $T_*^{(1)}$. The class $v_2 b$ in $T_*^{(1)}$ is represented by the non-zero class $t\mu \cdot b_1$ in $E^\infty(S^1)$ in filtration -2, and we deduce that $b^{p-1} \in T_*^{(0)}$ must be represented by $-t\mu$ in $E^\infty(S^1)$. It follows that if a class $x \in T_{2p+3}^{(1)}$ is represented by $b_1 \cdot x_n$, then $b^{p-2}x$ is represented by $-t\mu \cdot x_n$ in 2 filtration degrees lower. Using a coarser filtration that ignores this shift, and considering our formulas for $E^\infty(S^1)_{2p+3}^{(1)}$ and $E^\infty(S^1)_{2p^2-1}^{(0)}$ given above, we deduce claim (1) from the corresponding claim for the associated graded, with z represented by $b_1 \cdot x_0$.

To prove claim (2), we notice that if a class $y \in T_{2p+3}^{(1)}$ is represented by $b_1 \cdot x_n$ with $n \geq 1$, then $R_*(y)$ will be represented by $b_1 \cdot x_{n-1}$ in higher filtration, up to some non-zero scalar multiple: this follows directly from Lemmas 7.1 and 7.2. In particular, $R_*(y) \neq y$. This implies the following claim:

(3) The group $\ker(R-1)_{2p+3}^{(1)}$ contains at most one class represented by $b_1 \cdot x_0$.

Now consider the class $\tilde{x}_0 = \widetilde{\lambda_1 t^{p-1}} \in T_1^{(0)}$ given in Definition 7.6. By definition, this class lies in $\ker(R-1)_1^{(0)}$ and is represented by x_0. We also claim that

(4) The class $b\tilde{x}_0 \in \ker(R-1)_{2p+3}^{(1)}$ is not annihilated by b^{p-2}.

Since $b\tilde{x}_0$ is represented by $b_1 \cdot x_0$, claim (2) follows from claims (3) and (4).

Finally, to prove claim (4), we recall from [2, Theorem 8.2] that the class $v_2\tilde{x}_0 \in \ker(R-1)_{2p^2-1}^{(0)}$ is non-zero, and must be represented, in filtration degree lower then $-2p+2$, by a class in

$$\mathbb{F}_p\{t\mu \cdot x_n \,|\, n \geq 1\}.$$

None of these classes is annihilated by b_1. Therefore $bv_2\tilde{x}_0 = -b^p\tilde{x}_0$ is non-zero, and we deduce that $b\tilde{x}_0 \in \ker(R-1)^{(1)}_{2p+3}$ is not annihilated by b^{p-2}. □

Remark 7.5 The lemma above implies that $a_1 \in V(1)_*THH(ku_p)$ has a lift $a_1 \in V(1)_*K(ku_p)$ under the trace, with $b^{p-2}a_1 = \lambda_2$, see Theorem 8.1. It would be nice to have a more direct construction of such a lift. In fact, we conjecture that $a_1 \in V(1)_*K(ku_p)$ decomposes as bd, where $d \in V(1)_1K(KU_p)$ is a unit class, when mapped into $V(1)_*K(KU_p)$, see the discussion preceding Theorem 8.3 below.

Definition 7.6 We consider the following subgroups of $E^\infty(S^1)$:

$$A = E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu),$$

$$B_0 = E(\lambda_1) \otimes P_{p-2}(b_1) \otimes \mathbb{F}_p\{\mu^{-1}a_i, a_0 \,|\, 2 \le i \le p-1\},$$

$$\begin{aligned} B_k = &\Big(E(\lambda_1) \otimes P_{p-2}(b_1) \otimes \bigoplus_{0<d<p} \big(P_{r(k)-dp^{k-1}+1}(t\mu) \otimes \mathbb{F}_p\{a_1 t^{dp^{k-1}}\}\big)\Big) \\ &\oplus \Big(E(\lambda_1) \otimes \bigoplus_{0<d<p} P_{r(k)-dp^{k-1}}(t\mu) \otimes \mathbb{F}_p\{\lambda_2 t^{dp^{k-1}}\}\Big) \quad \text{for } k \ge 2 \text{ even}, \end{aligned}$$

$$\begin{aligned} B_k = &\ E(a_1) \otimes P_{p-1}(b_1) \\ &\otimes \bigoplus_{0<d<p} \big(P_{r(k)-dp^{k-1}}(t\mu) \otimes \mathbb{F}_p\{\lambda_1 t^{dp^{k-1}}\}\big) \quad \text{for } k \ge 1 \text{ odd}, \end{aligned}$$

and we let C be the span of the remaining monomials in $E^\infty(S^1)$. We then have a direct sum decomposition $E^\infty(S^1) = A \oplus B \oplus C$, with $B = \bigoplus_{k\ge 0} B_k$.

Lemma 7.7 *In dimensions larger than $2p-2$ there are closed subgroups $\tilde{A}$, $\tilde{B}_k$ and $\tilde{C}$ in $V(1)_*TF(ku_p)$, represented by A, B_k and C in $E^\infty(S^1)$ respectively, such that*

(a) *R_* restricts to the identity on $\tilde{A}$,*
(b) *R_* maps $\tilde{B}_{k+2}$ onto $\tilde{B}_k$ for $k \ge 0$,*
(c) *R_* maps $\tilde{B}_0$, $\tilde{B}_1$ and $\tilde{C}$ to zero.*

In these degrees $V(1)_*TF(ku_p) \cong \tilde{A} \oplus \tilde{B} \oplus \tilde{C}$, where $\tilde{B} = \prod_{k\ge 0} \tilde{B}_k$.

Proof On the associated graded $E^\infty(S^1)$, the homomorphism $(\Gamma_*\hat{\Gamma}_*^{-1})R_*^h$ has been described in Lemmas 7.1 and 7.2, and maps A isomorphically to itself, B_{k+2} onto B_k for $k \ge 0$, and B_0, B_1 and C to zero. It remains to find closed lifts of these groups in $V(1)_*TF(ku_p)$ with desired properties. We take $\tilde{A}$ to be the (closed) subalgebra of $V(1)_*TF(ku_p)$ generated by $\tilde{\lambda}_1$, $\tilde{a}_1$, b and

v_2. Then $\tilde{A}$ lifts A, by definition of its algebra generators and by the fact, proved above, that b^{p-1} is represented by $-t\mu$ in $E^\infty(S^1)$. Also, $\tilde{\lambda}_1$, b and v_2 are fixed under R_*, since they are in the image of tr_{F*}, and $\tilde{a}_1$ is fixed by definition. To construct $\tilde{B}_k$ for $k \geq 0$ and $\tilde{C}$, we follow the procedure given in [2, Theorem 7.7]. □

Definition 7.8 We denote $b \in V(1)_{2p+2}TC(ku_p)$ the image of the higher Bott element b, defined in 3.2, under the cyclotomic trace map

$$(\mathrm{trc})_* : V(1)_*K(ku_p) \to V(1)_*TC(ku_p).$$

Theorem 7.9 *The class $b \in V(1)_{2p+2}TC(ku_p)$ satisfies the relation*

$$b^{p-1} = -v_2.$$

There is an isomorphism of $P(b)$-modules

$$\begin{aligned} V(1)_*TC(ku_p) \cong\, & P(b) \otimes E(\partial, \lambda_1, a_1) \\ & \oplus P(b) \otimes E(a_1) \otimes \mathbb{F}_p\{t^d\lambda_1 \,|\, 0 < d < p\} \\ & \oplus P(b) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{u^i a_0, t^{p^2-p}\lambda_2 \,|\, 0 \leq i < p-2\}, \end{aligned}$$

where the degree of the classes is $|\partial| = -1$, $|\lambda_1| = 2p-1$, $|a_1| = 2p+3$, $|u^i a_0| = 2i+3$, $|\lambda_2| = 2p^2-1$ and $|t| = -2$.

Proof Recall that $TC(ku_p)$ is defined as the homotopy fiber of the map

$$R - 1 : TF(ku_p) \to TF(ku_p).$$

In $V(1)$-homotopy, it gives a short exact sequence of $P(v_2)$-modules

$$0 \to \Sigma^{-1}\mathrm{cok}(R-1)_* \to V(1)_*TC(ku_p) \to \ker(R-1)_* \to 0. \tag{7.1}$$

We have isomorphisms of $P(v_2)$-modules

$$\begin{aligned} \Sigma^{-1}\mathrm{cok}(R-1)_* &\cong \Sigma^{-1}\tilde{A} \\ \ker(R-1)_* &\cong \tilde{A} \oplus \lim_{k\geq 0 \text{ even}} \tilde{B}_k \oplus \lim_{k \geq 1 \text{ odd}} \tilde{B}_k. \end{aligned} \tag{7.2}$$

Indeed, $R_* - 1$ maps each factor of the decomposition $V(1)_*TF(ku_p) \cong \tilde{A} \oplus \tilde{B} \oplus \tilde{C}$ to itself. It restricts to zero on $\tilde{A}$ and to the identity on $\tilde{C}$. We have a short exact sequence

$$0 \to \lim_{k\geq 0 \text{ even}} \tilde{B}_k \to \prod_{k\geq 0 \text{ even}} \tilde{B}_k \xrightarrow{R_*-1} \prod_{k \geq 0 \text{ even}} \tilde{B}_k \to \lim_{k\geq 0 \text{ even}}{}^1 \tilde{B}_k \to 0,$$

and similarly for the $\tilde{B}_k$ with k odd. Here the limits are taken over the sequential system of maps $R_* : \tilde{B}_{k+2} \to \tilde{B}_k$ for $k \geq 0$ even or $k \geq 1$ odd. Since these maps are surjective, the $\lim^1$-terms are trivial. This proves our claims on $\Sigma^{-1} \operatorname{cok}(R-1)_*$ and $\ker(R-1)_*$ in (7.2).

For $k \geq 1$ odd, the group $\tilde{B}_k$ is isomorphic as a $P(v_2)$-module to a sum of $2(p-1)^2$ cyclic $P(v_2)$-modules

$$\tilde{B}^k \cong E(a_1) \otimes P_{r(k)}(v_2) \otimes P_{p-1}(b) \otimes \mathbb{F}_p\{\lambda_1 t^{dp^{k-1}} \mid 0 < d < p\}.$$

The map R_* respects this decomposition into cyclic $P(v_2)$-modules. Since the height of these modules grows to infinity with k, we deduce from the surjectivity of R_* that $\lim_{k \geq 1 \text{ odd}} \tilde{B}_k$ is a sum of $2(p-1)^2$ free cyclic $P(v_2)$-modules, given by an isomorphism

$$\lim_{k \geq 1 \text{ odd}} \tilde{B}_k \cong E(a_1) \otimes P(v_2) \otimes P_{p-1}(b) \otimes \mathbb{F}_p\{\lambda_1 t^d \mid 0 < d < p\}.$$

Similarly, for $k \geq 2$ even, $\tilde{B}_k$ is isomorphic to a sum of $2(p-1)^2$ cyclic $P(v_2)$-modules of height growing with k, and passing to the limit we have an isomorphism of $P(v_2)$-modules

$$\lim_{k \geq 0 \text{ even}} \tilde{B}_k \cong E(\lambda_1) \otimes P(v_2) \otimes P_{p-1}(b) \otimes \mathbb{F}_p\{a_1 t^{dp} \mid 0 < d < p\}.$$

Thus $\ker(R-1)_*$ is a free $P(v_2)$-module, and the exact sequence (7.1) splits. We have an isomorphism of $P(v_2)$-modules

$$\begin{aligned} V(1)_* TC(ku_p) \cong\, & P(v_2) \otimes P_{p-1}(b) \otimes E(\partial, \lambda_1, a_1) \\ & \oplus P(v_2) \otimes P_{p-1}(b) \otimes E(a_1) \otimes \mathbb{F}_p\{\lambda_1 t^d \mid 0 < d < p\} \\ & \oplus P(v_2) \otimes P_{p-1}(b) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{a_1 t^{pd} \mid 0 < d < p\} \end{aligned} \tag{7.3}$$

in degrees larger than $2p-2$, where the summand

$$P(v_2) \otimes P_{p-1}(b) \otimes E(\lambda_1, a_1) \otimes \mathbb{F}_p\{\partial\}$$

is the group $\operatorname{cok}(R-1)_* \cong \Sigma^{-1} \tilde{A}$. We now show that the relation

$$b^{p-1} = -v_2$$

holds in $V(1)_* TC(ku_p)$. Recall from Proposition 2.7 that the class $b'^{p-1} + v_2$ in $V(1)_{2p^2-2} K(\mathbb{Z}, 3)$ is annihilated by b'. This class maps by $\operatorname{trc}_* \phi_*$ to the class

$$b^{p-1} + v_2 \in V(1)_{2p^2-2} TC(ku_p),$$

which is therefore annihilated by b. Thus it suffices to show that zero is the only class in $V(1)_{2p^2-2}TC(ku_p)$ that is annihilated by b. We consider the short exact sequence

$$0 \to \mathrm{cok}(R-1)_{2p^2-1} \to V(1)_{2p^2-2}TC(ku_p) \to \ker(R-1)_{2p^2-2} \to 0$$

given in (7.1) above. Here

$$\ker(R-1)_* \subset V(1)_*TF(ku_p)$$

inherits via Γ_* the spectral sequence filtration of $V(1)_*THH(ku_p)^{hS^1}$. By (7.3), this filtration gives the short exact sequence

$$0 \to \mathbb{F}_p\{b^{p-2} \cdot \lambda_1 \cdot a_1 t^p\} \to \ker(R-1)_{2p^2-2} \to \mathbb{F}_p\{\overline{v_2}\} \to 0$$

in dimension $2p^2-2$, while in dimension $2p^2+2p$ it gives the short exact sequence

$$0 \to \mathbb{F}_p\{v_2 \cdot \lambda_1 \cdot a_1 t^p\} \to \ker(R-1)_{2p^2+2p} \to \mathbb{F}_p\{\overline{b \cdot v_2}\} \to 0.$$

Here $\overline{v_2}$ and $\overline{b \cdot v_2}$ are represented by $t\mu$ and $b_1 \cdot t\mu$ in $E^\infty(S^1)$, respectively. Multiplication with b is compatible with the filtration, and maps the former sequence to the latter one. First, notice that the class $\overline{v_2}$ maps to a non-zero class in $\mathbb{F}_p\{\overline{b \cdot v_2}\}$, since $b \cdot \overline{v_2}$ is represented by $b_1 \cdot t\mu$ in $E^\infty(S^1)$. Next, the relation $b^p = -bv_2$ in $\ker(R-1)_*$ implies

$$b^p \cdot \lambda_1 \cdot a_1 t^p = -v_2 \cdot b \cdot \lambda_1 \cdot a_1 t^p,$$

which is non-zero by (7.3). A fortiori $b^{p-1} \cdot \lambda_1 \cdot a_1 t^p \in \mathbb{F}_p\{v_2 \cdot \lambda_1 \cdot a_1 t^p\}$ is not zero either. Thus $\ker(R-1)_{2p^2-2}$ contains no non-zero class annihilated by b, and we deduce that

$$b^{p-1} + v_2 \in \partial(\mathrm{cok}(R-1)_{2p^2-1}) = \mathbb{F}_p\{b^{p-2} \cdot a_1 \cdot \partial\}.$$

However the class $b^{p-2} \cdot a_1 \cdot \partial$ is not annihilated by b, since by (7.3) we know that $b^p \cdot a_1 \cdot \partial = -v_2 \cdot b \cdot a_1 \cdot \partial$ is non-zero. This proves that $b^{p-1} + v_2$ must be zero.

In particular b is not a nilpotent class, and we have an isomorphism of $P(b)$-modules

$$\begin{aligned} V(1)_*TC(ku_p) \cong\, & P(b) \otimes E(\partial, \lambda_1, a_1) \\ & \oplus P(b) \otimes E(a_1) \otimes \mathbb{F}_p\{t^d \lambda_1 \mid 0 < d < p\} \\ & \oplus P(b) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{a_1 t^{pd} \mid 0 < d < p\} \end{aligned}$$

in degrees larger than $2p-2$. This proves that our formula for $V(1)_*TC(ku_p)$ is correct in dimensions greater than $2p-2$. Let us define M and N as

$$M = \bigoplus_{-1\le n\le 2p-2} V(1)_n TC(ku_p) \quad \text{and} \quad N = \bigoplus_{n\ge 2p-1} V(1)_n TC(ku_p).$$

We just argued that N is a free $P(b)$-module. We know by (5.3) that there is an isomorphism

$$M \cong \mathbb{F}_p\{\partial, 1, u^i a_0, \lambda_1 t^d, \partial\lambda_1 \,|\, 0\le i\le p-3,\ 1\le d\le p-1\}$$

of $\mathbb{F}_p$ -modules. This proves that the formula for $V(1)_*TC(ku_p)$ in Theorem 7.9 holds as an isomorphism of $\mathbb{F}_p$-modules. It only remains to show that for any non-zero class $m\in M$, we have $bm\neq 0$ in $V(1)_*TC(ku_p)$. By comparison with $V(1)_*TC(\ell_p)$ or with $V(1)_*THH(ku_p)^{hS^1}$, we know that either $m\lambda_1$ or mv_2 is non-zero. These products lie in N for degree reasons, so are not b-torsion classes. Therefore m is not a b-torsion class either. □

8 Algebraic K-theory

Theorem 8.1 *There is an isomorphism of $P(b)$-modules*

$$\begin{aligned} V(1)_*K(ku_p) \cong\, & P(b)\otimes E(\lambda_1, a_1)\oplus P(b)\otimes \mathbb{F}_p\{\partial\lambda_1, \partial b, \partial a_1, \partial\lambda_1 a_1\}\\ & \oplus P(b)\otimes E(a_1)\otimes \mathbb{F}_p\{t^d\lambda_1 \,|\, 0<d<p\}\\ & \oplus P(b)\otimes E(\lambda_1)\otimes \mathbb{F}_p\{\sigma_n, \lambda_2 t^{p^2-p} \,|\, 1\le n\le p-2\}\\ & \oplus \mathbb{F}_p\{s\},\end{aligned}$$

with $b^{p-1}=-v_2$. The degree of the generators is given by $|\partial|=-1$, $|\lambda_1| = 2p-1$, $|a_1|=2p+3$, $|\sigma_n|=2n+1$, $|t|=-2$, $|\lambda_2|=2p^2-1$ and $|s| = 2p-3$. The classes $1, \sigma_n, \lambda_1, b$ and a_1 map under the trace to $1, u^{n-1}a_0, \lambda_1$, b_1 and a_1 in $V(1)_*THH(ku_p)$, respectively, and the other given $P(b)$-module generators map to zero.

Proof There is a cofibre sequence of spectra [22]

$$K(ku_p)_p \to TC(ku_p) \to \Sigma^{-1}H\mathbb{Z}_p \to \Sigma K(ku_p)_p.$$

We have an isomorphism $V(1)_*\Sigma^{-1}H\mathbb{Z}_p \cong \mathbb{F}_p\{\partial, \epsilon\}$ with a primary v_1 Bockstein $\beta_{1,1}(\epsilon)=\partial$. Here ∂ is the image of the class $\partial\in V(1)_{-1}TC(ku_p)$, while ϵ maps by the connecting homomorphism to a class

$$s\in V(1)_{2p-3}K(ku_p).$$

These facts, together with Theorem 7.9, allow us to establish our formula for $V(1)_*K(ku_p)$. The statement on the trace follows from the definition of the given $P(b)$-module generators. □

The following corollary is a restatement of Proposition 1.2 part (b) of the introduction.

Corollary 8.2 *There is a short exact sequence of $P(b)$-modules*

$$0 \to K \to P(b) \otimes_{P(v_2)} V(1)_*K(\ell_p) \xrightarrow{\mu} K(ku_p) \to Q \to 0$$

where K and Q are finite (and hence torsion) $P(b)$-modules given by

$$\begin{aligned} K &= \mathbb{F}_p\{b^k a \mid 1 \le k \le p-2\}, \quad \textit{and} \\ Q &= P_{p-2}(b) \otimes \mathbb{F}_p\{\partial b, \partial a_1, a_1, \partial\lambda_1 a_1, \lambda_1 a_1\} \\ &\quad \oplus P_{p-2}(b) \otimes \mathbb{F}_p\{a_1\lambda_1 t^d \mid 0 < d < p\} \\ &\quad \oplus E(\lambda_1) \otimes \mathbb{F}_p\{\sigma_n b^{i_n} \mid 1 \le n \le p-2,\ 0 \le i_n \le p-2-n\}. \end{aligned}$$

Here $a \in V(1)_{2p-3}K(\ell_p)$ is the class annihilated by v_2 and mapping to s. In particular we have an isomorphism $P(b, b^{-1})$-algebras

$$P(b, b^{-1}) \otimes_{P(v_2)} V(1)_*K(\ell_p) \cong V(1)_*K(ku_p)[b^{-1}].$$

Proof This follows from the formulas for $V(1)_*K(\ell_p)$ and for $V(1)_*K(ku_p)$ given in [2, Theorem 9.1] and Theorem 8.1, and the fact that $V(1)_*K(\ell_p)$ includes as the summand of δ-weight zero in $V(1)_*K(ku_p)$, see [1, Theorem 10.2]. Notice that for $1 \le d \le p-2$ the class

$$\lambda_2 t^{dp} \in V(1)_{2p^2-pd-1}K(\ell_p)$$

maps to $\sigma_d b^{p-1-d}$, up to a non-zero scalar multiple. □

Blumberg and Mandell [8] have proved a conjecture of John Rognes that there is a localization cofibre sequence

$$K(\mathbb{Z}_p) \xrightarrow{\tau} K(ku_p) \xrightarrow{j} K(KU_p) \to \Sigma K(\mathbb{Z}_p),$$

relating the algebraic K-theory of ku_p, of its localization $KU_p = ku_p[u^{-1}]$ (i.e. periodic K-theory), and of its mod (u) reduction $H\mathbb{Z}_p$. The $V(1)$-homotopy of $K(\mathbb{Z}_p)$ and $K(ku_p)$ is known, but we need to compute also the transfer map τ_* and solve a $P(b)$-module extension if we seek a decent description of $V(1)_*K(KU_p)$. Let us therefore assume that this localization sequence maps via trace maps to a corresponding localization sequence in topological Hochschild homology, building a homotopy commutative diagram of horizontal fibre sequences

$$\begin{tikzcd}
K(\mathbb{Z}_p) \arrow[r, "\tau"] \arrow[d, "\mathrm{tr}"] & K(ku_p) \arrow[r, "j"] \arrow[d, "\mathrm{tr}"] & K(KU_p) \arrow[r] \arrow[d, "\mathrm{tr}"] & \Sigma K(\mathbb{Z}_p) \arrow[d, "\Sigma\mathrm{tr}"] \\
THH(\mathbb{Z}_p) \arrow[r, "\tau"] & THH(ku_p) \arrow[r, "j"] & THH(ku_p|KU_p) \arrow[r] & \Sigma THH(\mathbb{Z}_p),
\end{tikzcd} \tag{8.1}$$

as conjectured by Lars Hesselholt, compare with Remark 8.4 below. The $V(1)$-homotopy of the bottom line was described in [1, Sect. 10]. The $V(1)$-homotopy groups of $K(\mathbb{Z}_p)$ are given by an isomorphism [22]

$$V(1)_*K(\mathbb{Z}_p) \cong E(\lambda_1) \oplus \mathbb{F}_p\{\partial v_1, \partial\lambda_1\} \oplus \mathbb{F}_p\{\lambda_1 t^d \mid 0 < d < p\}.$$

The class ∂v_1 maps to s in $V(1)_*K(ku_p)$ via τ_*. The class $1 \in V(1)_0K(\mathbb{Z}_p)$ is in the kernel of τ_*, because it is v_2-torsion and there is no torsion class in $V(1)_0K(ku_p)$. Let $d \in V(1)_1K(KU_p)$ be the class mapping to $1 \in V(1)_0K(\mathbb{Z}_p)$ via the connecting homomorphism. Presumably d corresponds to the added unit or the self-equivalence

$$KU_p \xrightarrow{u} \Sigma^{-2}KU_p \xrightarrow{\simeq} KU_p,$$

where u denotes multiplication by the Bott class, and the second map is the Bott equivalence. The class d maps in $V(1)_1THH(ku_p|KU_p)$ to a class with the same name. In [1, Sect. 10] we establish an (additive) isomorphism

$$V(1)_*THH(ku_p|KU_p) \cong P_{p-1}(u) \otimes E(d, \lambda_1) \otimes P(\mu_1). \tag{8.2}$$

If this is an isomorphism of algebras, then the relation $j_*(b_1)d = j_*(a_1)$ holds in $V(1)_*THH(ku_p|KU_p)$, and it lifts to the relation $j_*(b)d = j_*(a_1)$ in $V(1)_*K(KU_p)$. By inspection this determines the structure of $V(1)_*K(KU_p)$ as a $P(b)$-module.

Theorem 8.3 *Under the hypothesis that there exists a commutative diagram of localization sequences* (8.1), *and that the isomorphism* (8.2) *is one of al-*

gebras, we have an isomorphism of $P(b)$-modules

$$\begin{aligned} V(1)_*K(KU_p) \cong P(b) \otimes E(\lambda_1, d) &\oplus P(b) \otimes \mathbb{F}_p\{\partial\lambda_1, \partial b, \partial a_1, \partial\lambda_1 d\} \\ &\oplus P(b) \otimes E(d) \otimes \mathbb{F}_p\{t^d\lambda_1 \,|\, 0 < d < p\} \\ &\oplus P(b) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{\sigma_n, \lambda_2 t^{p^2-p} \,|\, 1 \le n \le p-2\}. \end{aligned}$$

The class d has degree 1, *and the other classes have the degree given in Theorem* 8.1.

Remark 8.4 Consider a complete discrete valuation field K of characteristic zero with perfect residue field k of characteristic $p \geq 3$, and let A be its valuation ring. Hesselholt and Madsen [23] compute the $V(0)$-homotopy of $K(A)$ and $K(K)$ by means of the cyclotomic trace. They introduce a relative version of topological cyclic homology, denoted $TC(A|K)$, that sits in a localization cofibre sequence

$$TC(k) \to TC(A) \to TC(A|K) \to \Sigma TC(k).$$

The computation of $V(0)_*TC(A|K)$ is achieved by using the rich algebraic structure on the $V(0)$-homotopy groups of the tower $TR^\bullet(A|K)$, and described in terms of the de Rham-Witt complex with log poles $W_\bullet\omega^*(A, A \cap K^\times)$, see [23, Theorem C]. Then $V(0)_*TC(A)$ can be evaluated by means of the localization sequence. This approach has, in particular, the advantage of avoiding a computation of $V(0)_*TR^\bullet(A)$, which seems quite intractable.

Continuing the discussion in [1, Sect. 10] on a relative trace for ku_p, and following Lars Hesselholt, one could speculate on the existence of a relative term $TC(ku_p|KU_p)$ fitting in a localization sequence

$$TC(H\mathbb{Z}_p) \to TC(ku_p) \to TC(ku_p|KU_p) \to \Sigma TC(H\mathbb{Z}_p),$$

through which the trace of diagram (8.1) factorizes. By analogy with the case of complete discrete valuation fields, we expect that a computation of $V(1)_*TR^n(ku_p|KU_p)$ should be easier to handle than the computation of $V(1)_*TR^n(ku_p)$ presented in this paper. In fact, the advantage of such an approach is already apparent when comparing

$$V(1)_*TR^1(ku_p|KU_p) = V(1)_*THH(ku_p|KU_p)$$

in (8.2) with $V(1)_*THH(ku_p)$ in (4.1), and is also confirmed by partial, hypothetical computations of $V(1)_*TR^n(\ell_p|L_p)$ and $V(1)_*TR^n(ku_p|KU_p)$ by Lars Hesselholt (private communication) and the author.

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