I Homotopy groups

We denote Top the category of spaces and continuous maps;
The set of continuous maps from X to Y will be denoted $\mathrm{Top}(X,Y)$

1.1 Def For $f, g \in \mathrm{Top}(X,Y)$, a homotopy from f to g is a map $H: X \times I \to Y$, $I = [0,1] \subset \mathbb{R}$, such that

$$\begin{tikzcd} X \arrow[r, hook, "i_0"] \arrow[dr, bend right, "f"'] & X \times I \arrow[d, "H"] & X \arrow[l, hook', "i_1"'] \arrow[dl, bend left, "g"] \\ & Y & \end{tikzcd}$$

commutes; here $i_t : X \to X \times I$, $x \mapsto (x,t)$

We denote this by $f \overset{H}{\simeq} g$. We say that f and g are homotopic (denoted $f \simeq g$) if $\exists H$, $f \overset{H}{\simeq} g$. Let $H_t = H \circ i_t$.

1.2 Lemma The relation $\simeq$ on $\mathrm{Top}(X,Y)$ is an equivalence rel.
This relation is compatible with composition in the sense that

$$\underset{\substack{\rotatebox{90}{$\in$} \\ g,\, g'}}{\mathrm{Top}(Y,Z)} \times \underset{\substack{\rotatebox{90}{$\in$} \\ f,\, f'}}{\mathrm{Top}(X,Y)} \xrightarrow{\circ} \mathrm{Top}(X,Z) \qquad f \simeq f' \text{ and } g \simeq g' \Rightarrow g \circ f \simeq g' \circ f'.$$

proof: let $f, g, h \in \mathrm{Top}(X,Y)$.
By $X \times I \xrightarrow{H} Y$, $H(x,t) = f(x)$ $\forall (x,t)$, we see $f \overset{H}{\simeq} f$.
If $f \overset{K}{\simeq} g$, then $g \overset{L}{\simeq} f$ by $L: X \times I \to Y$, $L(x,t) = K(x, 1-t)$
If $f \overset{K}{\simeq} g \overset{L}{\simeq} h$, then $f \overset{K*L}{\simeq} g$ where $K*L : X \times I \to Y$ is given by

$$K*L(x,t) = \begin{cases} k(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ L(x, 2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Thus $\simeq$ is an equivalence relation on $\mathrm{Top}(X,Y)$.
For the compatibility with composition, given $f \overset{F}{\simeq} f'$ and $g \overset{G}{\simeq} g'$, we use transitivity $g \circ f \overset{K}{\simeq} g' \circ f \overset{L}{\simeq} g' \circ f'$ for
$K : X \times I \xrightarrow{f \times id} Y \times I \xrightarrow{G} Z$, $L : X \times I \xrightarrow{F} Y \xrightarrow{g'} Z$. ▨

1.3 Def We define $\mathrm{ho}(\mathrm{Top})$, the homotopy category of spaces by $\mathrm{Ob}(\mathrm{ho}(\mathrm{Top})) = \mathrm{Ob}(\mathrm{Top})$, $\mathrm{hoTop}(X,X] := [X,Y] := \mathrm{Top}(X,Y)/_{\simeq}$.
The class of $f \in \mathrm{Top}(X,Y)$ in $[X,Y]$ is denoted $[f]$, and

composition is defined by $[f]\circ[g] := [f\circ g]$ (well defined by 1.2). The identity of $X \in hoTop$ is $[id_X]$.

Remark : We have an obvious functor $Top \to hoTop$.

1.4 Def A map $f: X \to Y$ is called a homotopy equivalence if $[f] \in hoTop(X,Y)$ is an isomorphism.

We say that spaces X, Y are homotopy equivalent, (denoted $X \simeq Y$), if they are isomorphic in hoTop. We say that X is contractible if $X \simeq \{*\}$ (one-point space).

1.5 Example : For $n \in \mathbb{N}$, any two maps $f, g : X \to \mathbb{R}^n$ are homotopic : $f \overset{H}{\simeq} g$ with $H: X \times I \to \mathbb{R}^n$, $H(x,t) = (1-t)f(x) + t g(x)$.

Thus $[X, \mathbb{R}^n]$ has a unique element!

In particular, $\{0\} \hookrightarrow \mathbb{R}^n$ is a homotopy equivalence, thus $\mathbb{R}^n$ is contractible.

1.6 Variants : (1) Let Top^2 be the category of pairs of spaces : $Ob(Top^2)$ consists of pairs (X, A) with $X, A \in Top$, $A \subset X$ subspace.
$Top^2((X,A),(Y,B)) = \{ f \in Top(X,Y) \,;\, f(A) \subset B \}$, with the obvious composition.

The category $hoTop^2$ is defined in the same way, using the following notion of homotopy : For $f, g \in Top^2((X,A),(Y,B))$, a homotopy $f \overset{H}{\simeq} g$ is a map $H : X \times I \to Y$ where $H_t : X \to Y$, $x \mapsto H(x,t)$ is a map of pairs ($H_t(A) \subset B$) for all $t \in I$.

(2) Let Top_* be the category of pointed spaces : its objects are pairs (X, x_0) with $x_0 \in X$ a closed point. A map $f : (X, x_0) \to (Y, y_0)$ is requested to satisfy $f(x_0) = y_0$, idem for homotopies ($H(x_0, t) = y_0 \;\; \forall\, t \in I$).

We get hoTop$_*$, the homotopy cat of pointed spaces.

(3) Top$^2_*$ and hoTop$^2_*$: pointed pairs (X, A, a_0), $a_0 \in A \subset X$.

1.7 Remarks: (1) Homotopy theorists do not work with hoTop. For eg, to show that $f: X \to Y$ is a htpy equivalence, one needs to exhibit maps $g: Y \to X$ and H, K with $f \circ g \overset{H}{\simeq} id_Y$, $g \circ f \overset{K}{\simeq} id_X$
But maps are difficult to produce!

A solution:

Instead, we want a derived (or localized) category Top$_*[W^{-1}]$ obtained by inverting a class of morphisms that we want to be the "right" notion of equivalence.
We will do this for the class of weak equivalences: those maps that induce an isomorphism on *homotopy groups*.

(2) hoTop (and in gen. Top$[W^{-1}]$) are not (co)complete, so constructions are difficult in these categories.

Solution: do not perform these construction in Top$[W^{-1}]$, but in "a model" for Top$[W^{-1}]$ with homotopy meaningfull constructions:

- model categories (Quillen) } lecture
- quasicategories or $(\infty,1)$-categories } Homotopy II.

(3) We want to use *exact sequences* in an optimal setting triangulated categories. But Top$[W^{-1}]$ is only "semi"-triangulated.

Solution: Work in stable homotopy theory, based on spectra. In this lecture series we will motivate the introduction of spectra.

1.8 Def For $n \in \mathbb{N}$, let

$$I^n = \begin{cases} \{0\} & n=0 \\ \underbrace{I \times \ldots \times I}_{n \text{ fact.}} & \text{if } n \geq 1, \end{cases} \qquad \partial I^n = \begin{cases} \emptyset & n=0 \\ \{(t_i)_i \in I^n ;\ \exists i, t_i \in \{0,1\}\} \end{cases}$$

For $(X, x_0) \in Top_*$, let

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)] \quad \text{in } hoTop^2.$$

For $n \geq 1$ and $1 \leq i \leq n$, let $+_i$ be the bin. operation on $\pi_n(X, x_0)$ given by $[f] +_i [g] := [f +_i g]$, where

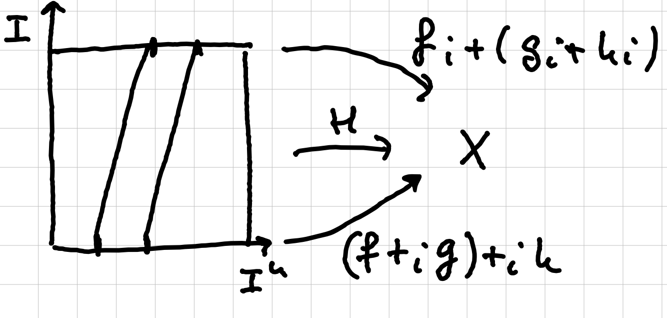
$$(f +_i g)(t_1, \ldots, t_n) = \begin{cases} f(t_1, \ldots, t_{i-1}, 2t_i, t_{i+1}, \ldots, t_n) & t_i \leq \frac{1}{2} \\ g(t_1, \ldots, t_{i-1}, 2t_i - 1, t_{i+1}, \ldots, t_n) & \frac{1}{2} \leq t_i. \end{cases}$$

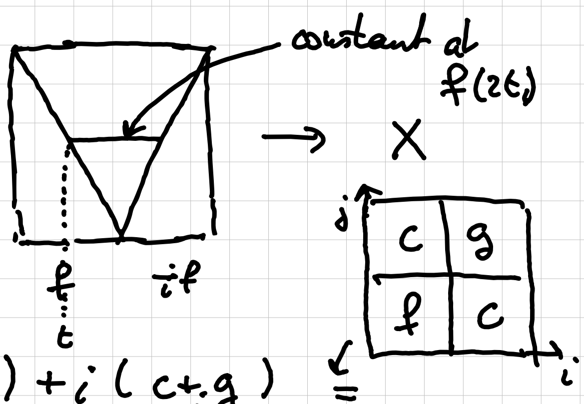
1.9 Lemma For $n \geq 1$, $1 \leq i \leq n$, $+_i$ defines a group structure on $\pi_n(X, x_0)$, with neutral él $0 = [c]$, $c : I^n \to X$, $t \mapsto x_0$, and inverse $-_i[f] = [-_i f]$, $-_i f(t_1, \ldots, t_n) =$

$$f(t_1, \ldots, t_{i-1}, 1 - t_i, t_{i+1}, \ldots, t_n).$$

Moreover, if $n \geq 2$, $1 \leq i, j \leq n$, then $+_i = +_j$ on $\pi_n(X, x_0)$, and this operation is commutative.

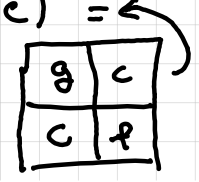
proof: that $(\pi_n(X, x_0), +_i)$ is a group for $n \geq 1$ is essentially the same proof than for π_1 :





$+_i = +_j$: $f +_i g \simeq (f +_j c) +_i (c +_j g) \overset{\swarrow}{=}$
$(f +_i c) +_j (c +_i g) \simeq f +_j g$, thus $[f] +_i [g] = [f] +_j [g]$.

Moreover $f +_1 g \simeq f +_2 g \simeq (c +_1 f) +_2 (g +_1 c) =$
$(c +_2 g) +_1 (f +_2 c) \simeq g +_1 f$. ▨



1.10 Def: For $(X, x_0) \in \mathrm{Top}_*$, (called the *fundamental group*)

- $\pi_0(X, x_0)$ (is a pointed set)
- $(\pi_1(X, x_0), \bullet = +_1)$ (is a group, noted multiplicatively
- $(\pi_n(X, x_0), + = +_1)$, $n \geq 2$ (is an abelian group)

called the *n-th homotopy group* of (X, x_0).

> Note that by definition, the functor $\pi_n : \mathrm{Top}_* \longrightarrow \begin{cases} \mathrm{Set}_* & n=0 \\ \mathrm{Gp} & n=1 \\ \mathrm{Ab} & n\geq 2 \end{cases}$
> factors through $\mathrm{Top}_* \to \mathrm{hoTop}_*$.
> We also call $\pi_n : \mathrm{hoTop}_* \to \{ \vdots$ the nth homotopy group

Question: Can the collection $\{\pi_n\}_{n\in\mathbb{N}}$ be promoted to a "homology theory"?

We have homotopy invariance. Let us now discuss exactness.
We need to define the homotopy groups of a pair (X, A).

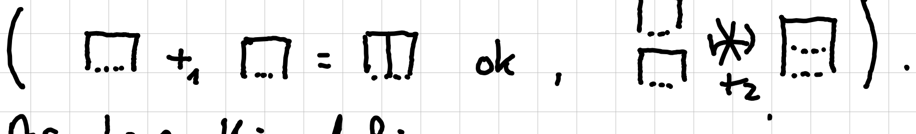
1.11. Def For $n \geq 1$, let $J^{n-1} \subset \partial I^n$ be defined as

$$J^{n-1} = \begin{cases} \{1\} & \text{if } n=1 \\ (\partial I^{n-1} \times I) \cup (I^{n-1} \times \{1\}) \subset \partial I^n \subset I^n & n \geq 2 \end{cases}$$

For $(X, A, *) \in \mathrm{Top}^2_*$, using hoTop^3, we define

$$\pi_n(X, A, *) = [(I^n, \partial I^n, J^{n-1}), (X, A, \{a_0\})].$$

If $n \geq 2$ and $1 \leq i \leq \underline{\underline{n-1}}$, define the binary operation $+_i$ on $\pi_n(X, A, *)$ by $[f] +_i [g] = [f +_i g]$,
$\uparrow$ as above.



As above, this defines

- a pointed-set structure on $\pi_1(X, A, a_0)$;

- a group structure on $\pi_2(X, A, a_0)$ (with $+ = +_1$) ;
- an (abelian) group structure on $\pi_3(X, A, a_0)$, with $+ = +_1 = +_i$ ($\forall\ 1 \leq i \leq n-1$).

We obtain a functor

$$\pi_n : (\text{ho})\text{Top}^2 \longrightarrow \begin{cases} \text{Sets}_* & n=1 \\ \text{Gp} & n=2 \\ \text{Ab} & n \geq 3 \end{cases}$$

called the <u>n-th relative homotopy group</u> (of a pointed pair)

<u>1.12 Lemma + Def.</u>: Let $(X, A, a_0) \in \text{Top}^2_*$. Let $d : I^{n-1} \to I^{n-1} \times \{0\} \subset I^n$ be the inclusion. For $n \geq 1$, let $\pi_n(X, A, a_0) \xrightarrow{\partial_n} \pi_{n-1}(A, a_0)$ given by $\partial_n([f]) = [f \circ d]$. Then ∂_n is a well defined morphism of pointed sets for $n=1$ and groups for $n \geq 2$, and defines a natural transformation of functors $\partial_n : \pi_n \to \pi_{n-1} \circ r : \text{Top}^2_* \to \begin{cases} \text{Sets} \\ \text{Gp} \\ \text{Ab} \end{cases}$

where $r : \text{Top}^2_* \to \text{Top}_*$, $r(X, A, a_0) = (A, a_0)$.
We call ∂_n <u>the connecting homomorphism</u>.

<u>proof</u>: obvious ... $\blacksquare$

<u>1.13 Theorem</u> (Exact homotopy sequence). Let $(X, A, a_0) \in \text{Top}^2_*$ and $i : (A, a_0) \hookrightarrow (X, a_0)$, $j : (X, a_0, a_0) \to (X, A, a_0)$ be the inclusions. Then the sequence

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+1}"] & \pi_n(A, a_0) \arrow[r] & \pi_n(X, a_0) \arrow[r] & \pi_n(X, A, a_0) \arrow[dll, "\partial_n"'] \\
& \pi_{n-1}(A, a_0) \arrow[r] & \cdots \arrow[r] & \pi_1(X, A, a_0) \arrow[dll, "\partial_1"'] \\
& \pi_0(A, a_0) \arrow[r] & \pi_0(X, a_0) &
\end{tikzcd}$$

is exact (and natural in (X, A, a_0))

<u>Note</u>: a sequence $(S, s) \xrightarrow{\alpha} (T, t) \xrightarrow{\beta} (U, u)$ of pointed sets is exact at (T, t) if $\text{Im}(\alpha) = \beta^{-1}(\{u\})$.

Proof : sketch on blackboard ; exercice ! ▨

How does $\pi_n(X, x_0)$ depend on the choice of $x_0 \in X$?

We introduce first the fundamental Groupoid

Recall : a groupoid is a small category in which all morph. are iso's.

1.14 Definition let $X \in Top$. Define the small cat ΠX :

$Ob(\Pi X)$ = underlying set of X ; For $a, b \in X$,

$\Pi X(a,b) = \Omega(X,a,b)/\sim$ where $\Omega(X,a.b) =$

$\{f : I \to X ; f(0) = a, f(1) = b\}$ and $\sim \; = \overset{H}{\simeq}$ with $H_t \in \Omega(X,a,b)$ $\forall t$.

$id_a = [C_a]$, $C_a : I \to X$ constant with value a.

Composition : $\Pi X(b,c) \times \Pi X(a,b) \xrightarrow{\circ} \Pi(a,c)$ ⚠

$([\beta], [\alpha]) \qquad [\beta] \circ [\alpha] = [\alpha * \beta]$

where $*$ is the usual concatenation of paths (as $+_1$ in 1.8).

We call ΠX the fundamental groupoid of X.

Remark : checking this is well defined and forms a category is an easy exercice (analogue to proving that $\pi_1(X,a)$ is a group).

ΠX is obviously a groupoid, with $[\alpha]^{-1} = [\alpha^{-1}]$, where $\alpha^{-1} : I \to X$, $t \mapsto \alpha(1-t)$ (reverse path).

1.15 Definition : if G is a groupoid, $a \in Ob(G)$, let

- $\pi_0(G) = Ob(G)/\sim$, with $a \sim b \iff G(a,b) \neq \emptyset$.
- $\pi_1(G,a) = Aut_G(a)$.

Note that by definition, for $(X, x_0) \in Top_*$,

$\pi_0(\Pi X) = \pi_0(X, x_0)$ (as unpointed sets)

$\pi_1(\Pi X, x_0) = \pi_1(X, x_0)$ as groups.

1.16 Functoriality of Π :

(a) A map $f : X \to Y$ extends to a functor $\Pi X \xrightarrow{f_*} \Pi Y$

(b) A homotopy $f \overset{H}{\simeq} g : X \to Y$ defines a natural isomorphism $H_* : f_* \to g_*$: indeed, for $[\alpha] \in \Pi X(a,b)$,

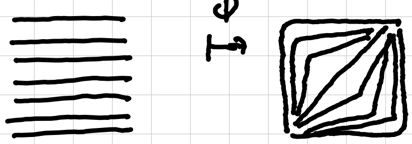
$$\begin{tikzcd} f(a) \arrow[r, "f_*{[\alpha]}"] \arrow[d, "H_a"'] & f(b) \arrow[d, "H_b"] \\ g(a) \arrow[r, "g_*{[\alpha]}"'] & g(b) \end{tikzcd}$$

$H_a = [\, I \to Y,\ t \mapsto H(a,t)]$

Have to check it commutes:

$$H_b * (f \circ \alpha) \overset{K}{\simeq} (g \circ \alpha) * H_b \ \text{rel}\ \{0,1\}$$

$$I \times I \xrightarrow{\phi} I \times I \xrightarrow{\alpha \times id} X \times I \xrightarrow{H} Y$$



1.17 Def

$X \in Top$, C a category. A local system (of objects in C) on X is a functor $F : \Pi X \to C$.

1.18 Lemma :

let $n \geq 0$ and

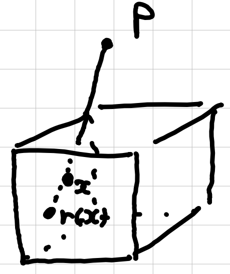
$$K_n = \partial I^n \times I \ \cup\ I^n \times \{0\} \ \subset\ \partial I^{n+1} \subset I^{n+1}$$

Then $K_n \overset{i}{\hookrightarrow} I^{n+1}$ admits a retraction $r : I^{n+1} \to K_n$.

In particular,

$$\begin{tikzcd} K_n \arrow[r, "f"] \arrow[d, "i"'] & X \\ I^{n+1} \arrow[ur, dotted, "\exists \tilde{f}"'] & \end{tikzcd}$$

proof: use a stereographic projection $r : I^{n+1} \to K_n$ from a point P above the center of the upper face of I^{n+1}

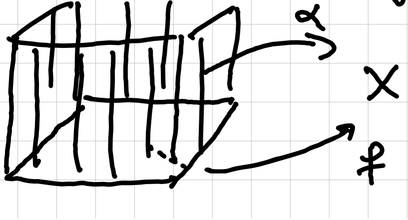


$\boxtimes$

In particular, if $\alpha \in \Omega(X,a,b)$ and $f : (I^n, \partial I^n) \to (X, \{a\})$, let $\tilde{f}_\alpha : I^{n+1} \to X$ be an extension of $\tilde{\tilde{f}}_\alpha : K^n \to X$

$$(s,t) \longmapsto \begin{cases} f(s) & \text{if } t = 0 \\ \alpha(t) & \text{if } s \in \partial I^n \end{cases}$$

(Here $s \in I^n$, $t \in I$, such that $(s,t) \in K_n$)

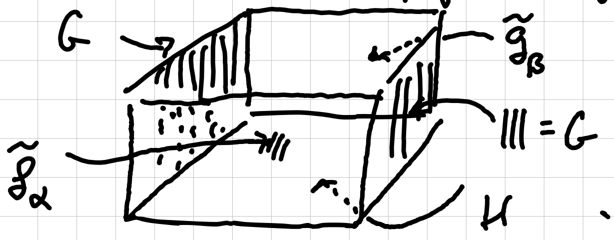


$f_\alpha : (I^n, \partial I^n) \to (X, b)$, $f_\alpha(s) = \tilde{f}_\alpha(s,1)$.

1.19 Proposition: For $X \in Top$ and $n \in \mathbb{N}$, we have a local system $\pi_n : \Pi X \longrightarrow \begin{cases} Sets & n=0 \\ gps & n=1 \\ Ab & n \geq 2 \end{cases}$ defined by

$$(a \xrightarrow{[\alpha]} b) \mapsto \pi_n(X,a) \longrightarrow \pi_n(X,b)$$
$$[f] \mapsto [f_\alpha]$$

proof: need to show first that $[f_\alpha]$ depends only on $[f]$ and $[\alpha]$. Easy exercise using lemma 1.18: if $f \overset{H}{\simeq} g$ and $\alpha \overset{G}{\simeq} \beta$, use $H, G, \tilde{f}_\alpha, \tilde{g}_\beta$ to define $K^{n+1} \to X$; Extend to $I^{n+1} \times I \to X$, so that "the top face" gives a homotopy $f_\alpha \simeq g_\beta$.



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we deduce:

- any path $x_0 \xrightarrow{u} x_0$ induces an iso $\pi_n(X,x_0) \to \pi_n(X,x_1)$
- $\pi_1(X,x_0)$ acts (on the right) on $\pi_n(X,x_0)$, $\forall n \geq 1$.
- Similarly, $\pi_1(A,a_0)$ acts on $\pi_n(X,A,a_0)$ $\forall n \geq 1$. We get that the long exact sequence of Thm 1.13 is one of right $\pi_1(A,a_0)$-groups/modules $(n \geq 1)$.

This action of $\pi_1(X,x_0)$ encodes the dependence on the base point of homotopy groups, (within a connected comp.). The space is called <u>simple</u> if this action is trivial $\forall x_0$.

1.20 Definition: A map $p: E \to B$ has the htpy lifting property (HLP) w.r.t. a space X if for any comm. square of the form given, a lift H exists:

$$\begin{tikzcd} X \arrow[r, "H_0"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dashed, "\exists H"] & B \end{tikzcd}$$

P is a Hurewicz fibration if it has the HLP w.r.t. to X $\forall X$

——— Serre ——————— " ——————— I^n $\forall n$.

<u>**1.21 Example**</u> : If $E = B\times F \xrightarrow{p} B$ is the projection, then p is a Hurewicz fibration :

$$\begin{tikzcd} X \arrow[r, "H_0"] \arrow[d, "i_0"'] & B\times F \arrow[d, "p"] \\ X\times I \arrow[r, "h"] \arrow[ur, "H"] & B \end{tikzcd}$$

If $q: E \to F$ is the other projection, take

$$H(x,t) = (\,h(x,t),\ qH_0(x))$$

<u>**1.22 Def**</u> : Given $P: (E, e_0) \to (B, b_0) \in Top_*$, $(F, e_0) = (\,P^{-1}(\{b_0\}), e_0)$ is called the fibre of P above b_0.

<u>**1.23 Prop**</u> Assume $(E, e_0) \to (B, b_0)$ is a Serre fibration, and (F, e_0) be the fibre above b_0. Then

$P_*: \pi_n(E, F, e_0) \longrightarrow \pi_n(B, b_0)$ is an iso $\forall n \geq 1$.

<u>proof</u> Replace $I^{n-1} \xhookrightarrow{i_0} I^n$ by a homeomorphic pair :

(i) P_* surjective: take $[f] \in \pi_n(B, b_0)$.

$$\begin{tikzcd} J^{n-1} \arrow[r, "c_{e_0}"] \arrow[d] & E \arrow[d, "p"] \\ I^n \arrow[r, "f"] \arrow[ur, dashed, "\bar f"] & B \end{tikzcd}$$

Then chose a lift $\bar f$. Then obviously $[\bar f] \in \pi_n(E, F, e_0)$, and $P_*[\bar f] = [f]$.

(ii) P_* is injective : Suppose

$f_0, f_1: (I^n, \partial I^n, J^{n-1}) \to (E, F, e_0)$ with

$P_*[f_0] = P_*[f_1]$. Chose $h: (I^n, \partial I^n)\times I \to (B, b_0)$ with $p\circ f_0 \overset{h}{\simeq} p\circ f_1$. Consider

$$\begin{tikzcd} L \arrow[r, "H_0"] \arrow[d, "i"'] & E \arrow[d] \\ I^{n+1} \arrow[r, "h"] \arrow[ur, "H"] & B \end{tikzcd}$$

Here L is $\partial I^{n+1} \setminus$ front face :

$$\begin{tikzcd}[column sep=small] L = & (I^n\times 0) \cup \arrow[dl, "f_0"] & (I^n\times 1) \cup \arrow[dll, "f_1"] & (J^{n-1}\times I) \arrow[dlll, "c_{e_0}"] \\ E & & & \end{tikzcd}$$

Then $f_0 \overset{H}{\simeq} f_1$

$\boxed{\text{Vd}}$

<u>**1.24 Corollary**</u> let $p: (E, e_0) \to (B, b_0)$ be a Serre fibr. and $(F, e_0) \xrightarrow{i} (E, e_0)$ the inclusion of the fibre.

Then we have the long htpy seq. of the "fibration sequence" $F \xrightarrow{i} E \xrightarrow{p} B$:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(F,e_0) \arrow[r, "i_*"] & \pi_n(E,e_0) \arrow[r, "p_*"] & \pi_n(B,b_0) \arrow[dll, bend left, "\partial_n"] \\
 & \cdots & & \\
{} \arrow[r, hook] & \pi_0(F,e_0) \arrow[r] & \pi_0(E,e_0) \arrow[r] & \pi_0(B,b_0)
\end{tikzcd}$$

Here ∂_n is defined as $\pi_n(B,b_0) \xrightarrow{p_*^{-1}} \pi_n(E,F,e_0) \xrightarrow{\partial_n} \pi_n(F,e_0)$

proof: combine 1.13 + 1.23. Exercise: exact at $\pi_0(E,e_0)$.

The next prop is useful for producing Serre fibrations: their lifting property is a "local" one.

1.25 Prop Suppose $p: E \to B$ in Top, and assume $\exists \{U_i\}_i$ open cover of B such that the restriction of p,

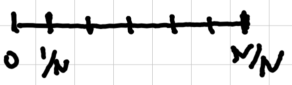
$p_i : p^{-1}(U_i) \to U_i$, is a Serre fibration for all i.

Then p is a Serre fibration.

proof: Suppose given a lifting problem :

$$\begin{tikzcd}
I^n \arrow[r, "H_0"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
I^n \times I \arrow[r, "h"] & B
\end{tikzcd}$$

By the Lemma of the Lebesgue Number, $\exists\, N \in \mathbb{N}$ such that $I^n \times I$ decomposes as a union of cubes W, products of one of the N sub-intervals



of I, for each factor I of I^{n+1}, such that $h(W) \subset U_i$ for some i.

Then one can lift h inductively on faces of the subcubes W, "layer by layer":

Induction : for $k = 0, 1, \dots, N$:

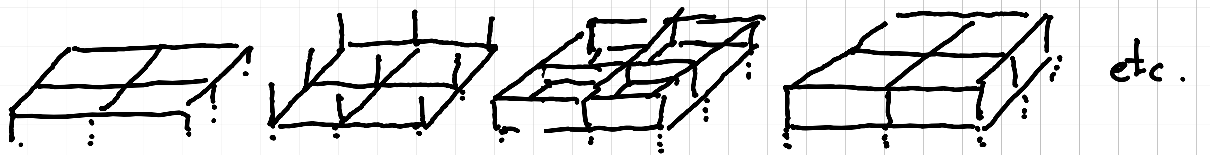
Prove that a lift $H : I^n \times [0, k \cdot \frac{1}{N}]$ exists.

$k=0$: This is given by H_0.

If lifted to $I^n \times [0, k \cdot \frac{1}{N}]$ for some $k < N$,

lift inductively for $l = 0, \dots, n$ along the l-dim'l faces of the sub cubes of $I^n \times \{k \cdot \frac{1}{N}\}$.

Sketch for $n = 2$, $N = \; = 2$



∎

<u>1.26 Example</u> : Fibre bundles are Serre Fibrations:
These are locally a projection, so are locally (hence globally) Serre fibrations.

<u>Recall</u> $E \xrightarrow{p} B$ is a fibre bundle with fibre F if there exists an open cover $\{U_i\}$ of B and a homéo $h_i : p^{-1}(U_i) \to U_i \times F$ for all i, such that

$$\begin{tikzcd} p^{-1}(U_i) \arrow[rr, "h_i"] \arrow[dr, "p"'] & & U_i \times F \arrow[dl, "pr_1"] \\ & U_i & \end{tikzcd}$$

commutes.

A nice example is given by "homogeneous" spaces.

<u>1.27 Proposition</u> : Let G be a topological group, and H be a subgroup of G. Let G/H be the space of H-orbits with the quotient topology. Let $p : G \to G/H$ the quotient map. Suppose that p has a "local section at e": $\exists$ U open nbhd of e in G/H and $s : U \to G$ with $ps = id_U$. Then for any closed subgroup $K \subset H$, the quotient map $q : G/K \to G/H$ is a fibre bundle with fibre H/K.

<u>proof</u> : Choose a local section $s : U \to G$ of p at e.
Define $\Phi : U \times H/K \longrightarrow G/K$, $([g]_H, [h]_K) \mapsto [s([g]_H) \cdot h]_K$
This is a well defined homeo onto its image $q^{-1}(U)$:
We have $q\,\Phi([g]_H, [h]_K) = q([s([g]_H) \cdot h]_K) =$
$[s([g]_H)]_H = p \circ s([g]_H) = [g]_H = pr_1([g]_H, [h]_K)$.

$$\begin{tikzcd} U \times H/K \arrow[r, "\phi"] \arrow[dr, "pr_1"'] & q^{-1}(U) \arrow[d, "q"] \arrow[dl, phantom, "\circlearrowleft"] \\ & U \end{tikzcd}$$

The inverse of ϕ is given by

$$\psi : q^{-1}(U) \longrightarrow U \times H/K$$

$$[g]_K \mapsto ([g]_H, [s([g]_H)^{-1} \cdot g]_K)$$

This proves that $q : G/K \to G/H$ is locally trivial at $[e]_H$.

To prove local triviality at $[x]_H \in G/H$ for $x \in G$:

$S' : xU \to G$, $\quad S'([g]_H) = x \cdot s([x^{-1}g]_H)$

is a local section of $p : G \to G/H$ at $[x]_H$ ▨

1.28 Example Let $n \in \mathbb{N}$, and consider

$G(n) = O(n)$ or $U(n)$ (orthogonal, unitary group).

If $0 \leq k \leq n$, the quotient map

$$p : G(n) \to G(n)/G(k)$$

admits a local section at $e = I_n$. Here $G(k)$ is viewed as

as a subgroup of $G(n)$ via $G(k) \to G(n)$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

To prove this: convenient to use the action of $G(n)$ on $\mathbb{F}^n$ ($= \mathbb{R}^n / \mathbb{C}^n$), in the next example.

1.29 Example Let $1 \leq \ell \leq n$, and

$$V_{n,\ell}^{\mathbb{F}} = \{ (v_1, \ldots, v_\ell) \in (\mathbb{F}^n)^\ell \; ; \; \langle v_i, v_j \rangle = \delta_{ij} \}$$

with the subspace topology: it is the space of <u>ℓ-frames</u> in $\mathbb{F}^n$, called the <u>Stiefeld manifold</u> $V_{n,\ell}^{\mathbb{F}}$.

Let $(e_1, \ldots, e_n)$ be the can. basis of $\mathbb{F}^n$, and take $x_0 = (e_{n-\ell+1}, \ldots, e_n)$ as base-point of $V_{n,\ell}$. Then $G(n)$ acts on $V_{n,\ell}$ diagonally and the stabilizer of x_0 is $G(n-\ell)$.

The action is continuous, transitive, and we get a homeo (compact to Hausdorff) $h_{n,\ell} : G(n)/_{G(n-\ell)} \to V_{n,\ell}$.

We get a commutative diagram with vertical homeos :

$$\begin{tikzcd}
G(n) \arrow[r, "p"] \arrow[d, "h_{n,n}"', "\cong"] & G(n)/G(n-\ell) \arrow[d, "h_{n,\ell}"', "\cong"] \\
V_{n,n} \arrow[r, "q"] & V_{n,\ell}
\end{tikzcd}$$

$$(v_1, \dots, v_n) \longmapsto (v_{n-\ell+1}, \dots, v_\ell).$$

To show that p has a local section at $[I_n]$ reduces to showing that q has a local section at x_0.

Take $\mathcal{U} = \{ (v_1, \dots, v_\ell) \in V_{n,\ell} \ ; \ (e_1, \dots, e_{n-\ell}, v_1, \dots, v_\ell)$ is a basis of $\mathbb{F}^n \}$, and $s : V_{n,\ell} \to V_{n,n}$ given by $s(v_1, \dots, v_\ell) = \mathrm{GSch}(e_1, \dots, e_{n-\ell}, v_1, \dots, v_\ell)$ (apply Gram-Schmidt starting with $v_\ell, v_{\ell-1}, \dots$). Then it is easy to check that s is a continuous section of q defined on $\mathcal{U}$ (which is an open nbhd of x_0).

$\Rightarrow$ p is a fibre bundle with fibre $G(n-\ell)$

From Proposition 1.27 we also get further examples:

We have fibre bundles p (resp q) with incl. of fibres i (resp j)

$$\begin{tikzcd}
G(n-k)/G(n-\ell) \arrow[r, "i"] \arrow[d, "h"] & G(n)/G(n-\ell) \arrow[r, "p"] \arrow[d, "h"] & G(n)/G(n-k) \arrow[d, "h"'] \\
V_{n-k,\ell-k} \arrow[r, "j"] & V_{n,\ell} \arrow[r, "q"] & V_{n,k}
\end{tikzcd}$$

$$(v_1, \dots, v_\ell) \longmapsto (v_{\ell-k+1}, \dots, v_\ell)$$

$\underline{1 \leq k \leq \ell \leq n}$

<u>1.30 Example</u>: Grassmann manifolds:

$G^{\mathbb{F}}_{n,\ell} = \{ V \subset \mathbb{F}^n \ ; \ V$ sub-vector space of dim $\ell \}$,

with the quotient topology of $V_{n,\ell} \xrightarrow{\pi} G_{n,\ell}$

$$(v_1, \dots, v_\ell) \mapsto \langle v_1, \dots, v_\ell \rangle.$$

sub-space spanned by $v_1, \dots, v_\ell$

If $x_0 = \langle e_{n-\ell+1}, \dots, e_n \rangle$, the stabilizer at x_0 of the obvious action of $G(n)$ on $G^{\mathbb{F}}_{n,\ell}$ is $G(n-\ell) \times G(\ell) \hookrightarrow G(n)$

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

We have a homeo: $G(n)/G(n-\ell) \times G(\ell) \xrightarrow{\ } G_{n,\ell}$.

Considering the inclusions $G(n-l) \hookrightarrow G(n-l) \times G(l) \to G(n)$, Proposition 1.27 gives us fibre bundles p, π with incl. of fibres i, resp. j:

$$\begin{tikzcd}
G(l) \arrow[r, "i"] \arrow[d, "\cong"'] & G(n)/G(n-l) \arrow[r, "p"] \arrow[d, "\cong"'] & G(n)/G(n-l)\times G(l) \arrow[d, "\cong"'] \\
V_{l,l} \arrow[r, "j"] & V_{n,l} \arrow[r, "\pi"] & G_{n,l}
\end{tikzcd}$$

1.31 Remark: The same construction works with $\mathbb{F} = \mathbb{H}$, the quaternions, leading to fibre bundles

$$V^{\mathbb{H}}_{n-k,\, l-k} \to V^{\mathbb{H}}_{n,l} \to V^{\mathbb{H}}_{n,k} \quad \text{and} \quad V^{\mathbb{H}}_{ll} \to V^{\mathbb{H}}_{n,l} \to G^{\mathbb{H}}_{n,l}.$$

1.32 Def: Taking $l=1$ above, we obtain the Hopf fibre bundles

$$\left.\begin{array}{l} S^0 \longrightarrow S^n \longrightarrow \mathbb{R}P^n \\ S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n \\ S^3 \longrightarrow S^{4n+3} \longrightarrow \mathbb{H}P^n \end{array}\right\} \text{ as } V^{\mathbb{F}}_{1,1} \to V^{\mathbb{F}}_{n+1,1} \to G^{\mathbb{F}}_{n+1,1} \text{ for } \mathbb{F} = \left\{\begin{array}{l} \mathbb{R} \\ \mathbb{C} \\ \mathbb{H} \end{array}\right.$$

and, for $n=1$, using $\mathbb{R}P^1 = S^1$, $\mathbb{C}P^1 = S^2$, $\mathbb{H}P^1 = S^4$,

$$S^0 \longrightarrow S^1 \xrightarrow{2} S^1, \quad S^1 \longrightarrow S^3 \xrightarrow{\eta} S^2, \quad S^3 \longrightarrow S^7 \xrightarrow{\nu} S^4$$

1.33 Remark the octonions $\mathbb{O}$, giving a non-associative, non-commutative algebra structure on $\mathbb{R}^8$.
We cannot define $\mathbb{O}P^k$ (lack of associativity) for $k \geq 3$, but $\mathbb{O}P^1 = S^8$ and $\mathbb{O}P^2$ exist! We get 2 more Hopf fibre bundles including

$$S^7 \longrightarrow S^{15} \xrightarrow{\sigma} S^8.$$ ☺

We now give another family of examples of fibrations. We want to use mapping spaces and have a cartesian closed category structure on Top. We won't give a complete def, but essentially: a closed cartesian structure on a cat C consists of the following properties / structures:

- admits (finite) cartesian products (in particular has a terminal object $* = \prod_{\emptyset}$).
- Set of morphisms are promoted to objects of C: we have a functor $C^{op} \times C \to C$, $(X,Y) \mapsto C(X,Y)$
- There is an isomorphism (natural in the 3 variables)

$$C(X\times Y, Z) \cong C(X, C(Y\times Z)) \quad \text{(adjunction)}$$

+ long list of axioms, eg $* \times X \cong X$, $C(*,X) \cong X$, etc.

Note: $C(X,Y)$ is then often denoted Y^X, and $Z^{X\times Y} \cong (Z^Y)^X$ is called the exponential law.

<u>1.34 Def</u> For $X, Y \in$ top, let $map(X,Y)$ be the space with underlaying set $Top(X,Y)$, and endowed with the <u>compact-open topology</u>: a pre-basis for this topology

consists of all subsets $Top^2((X,K),(Y,U))$ where K is compact (not nec. Hausdorff) in X and U open in Y.

<u>1.35 Lemma</u> The mapping space has the following properties:

(1) Y Hausdorff $\Rightarrow$ $map(X,Y)$ Hausdorff

(2) The map $map(X, Y\times Z) \to map(X,Y)\times map(X,Z)$ given by $f \mapsto (Pr_X \circ f, Pr_Y \circ f)$ is a homeo.

(3) We have a (cont) map $map(X\times Y, Z) \to map(X, map(Y,Z))$ given by $f \mapsto \left(\hat{f} : X \to map(Y,Z),\ \hat{f}(x) : \begin{array}{l} Y \to Z \\ y \mapsto f(x,y) \end{array}\right)$

(4) Suppose Y is Hausdorff and locally compact. Then, if $g \in map(X, map(Y,Z))$, $\check{g} : X\times Y \to Z$, $\check{g}(x,y) = g(x)(y)$ is continuous.

(5) If X Hausdorff and Y Hausdorff loc. compact, then the map

in (3) is a homeo, with inverse $g \mapsto \breve{g}$.

proof: exercice!

1.36 Remark: However, Top, with map as internal hom's, is not cartesian closed. Topologists have been looking for "good categories of topological spaces". One such is the category of "compactly generated weak Hausdorff" (CGWH) spaces, see May, Chap. 5 or Strickland.

This is cartesian closed, and mapping spaces are denoted $Y^X = \mathrm{map}(X, Y)$.

Note that there is a functor $k : \mathrm{Top} \to \mathrm{CGWH}$ and a natural transfo $k \overset{i}{\Rightarrow} \mathrm{Id}$.

If X is locally compact Hausdorff, then $k(X) \xrightarrow{i} X$ is a homeomorphism.

Beware: The forgetful functor $\mathrm{CGWH} \to \mathrm{Top}$ does not preserve limits or colimits in general!

1.37 Def: Let $f : X \to Y$ in Top. Let

$$F(f) = \{(x,w) \in X \times Y^I ;\ w(0) = f(x)\} \overset{\text{subspace}}{\subset} X \times Y^I,$$

or in other words we have a pull back square

$$\begin{tikzcd} F(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^I \arrow[d, "\varepsilon_0"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Let $i : X \longrightarrow F(f)$, $x \mapsto (x, c_{f(x)})$

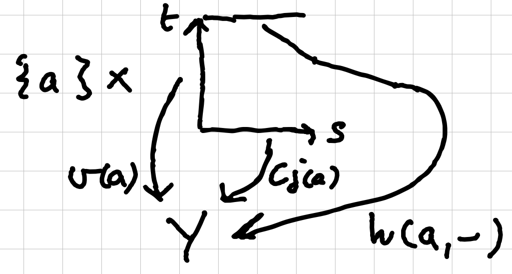
$\varepsilon_1 : F(f) \to Y$, $(x, w) \mapsto \omega(1)$

1.38 Lemma: The map $f : X \to Y$ factorizes as $\varepsilon_1 \circ i$; moreover, i is a homotopy equivalence and ε_1 has the HLP for nice spaces (in part. is a Serre Fibration).

proof: it is easy to check that the projection $F(f) \to X$, $(x, w) \mapsto x$ is a homotopy inverse of $i : X \to F(f)$.

$$\begin{tikzcd} A \arrow[r, "H_0"] \arrow[d, "i_0"'] & F(f) \arrow[d, "\varepsilon_1"] \\ A\times I \arrow[r, "w"'] \arrow[ur, dotted, "H"] & Y \end{tikzcd}$$

Consider a HL problem; say H_0 is given by $H_0(a) = (j(a), v(a))$ for $j: A \to X$, $v: A \to Y^I$. The lift we are looking for, $H: A\times I \to F(f)$, will be given as $H(a,s) = (J(a,s), V(a,s))$ for $J: A\times I \to X$ and $V: A\times I \to Y^I$. For J take $J(a,s) = j(a)$. We want $\check{V}: A\times I\times I \to Y$ to satisfy, for any $a\in A$,



Since we have a retraction  we can define $\check{V}$ satisfying the condition. ▨

We have useful variants of this construction.

<u>**1.39 Def**</u> Let A, B be subspaces of Y.

Let $F(A,Y,B) = \{ \omega \in Y^I ; \omega(0)\in A, \omega(1) \in B \}$ and let $\varepsilon_i : F(A,Y,B) \to B$, $\omega \mapsto \omega(i)$.

<u>**1.40 Lemma**</u>: ε_i is a (Serre) fibration

<u>proof</u>: if $B = Y$, and $i: A \to Y$ is the inclusion, then we have $F(A,Y,Y) = F(i)$, so that $F(A,Y,Y) \xrightarrow{\varepsilon} Y$ is a

$$\begin{tikzcd} {} \arrow[dr, "\varepsilon_1"'] & & {} \arrow[dl, "\varepsilon_1"] \\ & Y & \end{tikzcd}$$

fibration by Lemma 1.38.

$$\begin{tikzcd} F(A,Y,B) \arrow[r] \arrow[d, "\varepsilon_1"] & F(A,Y,Y) \arrow[d, "\varepsilon_1"] \\ B \arrow[r, hook] & Y \end{tikzcd}$$

Notice that we have a pull-back: so that the left ε_1 is also a fibration (see exercises). It holds also for ε_0 by symmetry. ▨

<u>**1.41 Def**</u> If $(X, x_0) \in Top_*$, then $F(\{x_0\}, X, \{x_0\})$ is called the <u>loop-space</u> of X and is denoted by $(\Omega X, *)$, based at $* = c_{x_0}$.

It is the fibre of the "path-loop" fibration:

$$(\Omega X, *) \hookrightarrow (F(x_0, X, X), *) \xrightarrow{\varepsilon_1} (X, x_0).$$

1.41 Corollary For $(X, x_0) \in Top_*$, we have a natural isomorphism $\pi_n(X, x_0) \xrightarrow{\partial_n} \pi_{n-1}(\Omega X, x_0)$ for all $n \geq 1$.

Proof: Use the long exact sequence of the path-loop fibration of $(X; x_0)$, together with the fact that $\{x_0\} \xhookrightarrow{i} F(x_0, X, X)$ is a weak equivalence.

1.42. Remark: A priori $\pi_1(X, x_0) \xrightarrow{\partial_1} \pi_0(\Omega X, x_0)$ is only an iso of sets! But in fact $(\Omega X, x_0)$ is a group "up to homotopy", i.e. we have a product $\Omega X \times \Omega X \to \Omega X$, $(u, v) \mapsto u * v$ satisfying the group axioms in $HoTop_*$.

This induces a group structure on $\pi_0(\Omega X, x_0)$, and ∂_1 gp. iso.

1.43 Def: Let $n \in \mathbb{N}$, $(X, A) \in Top^2$.

(1) X is called <u>n-connected</u> if $\forall\ 0 \leq k \leq n$, $\forall x_0 \in X$ $\pi_k(X, x_0) = 0$.

(2) X is called n-coconnected or n-truncated if $\forall x_0 \in X$, $\forall\ k > n$, $\pi_k(X, x_0) = 0$

(3) (X, A) is called n-connected if

$n = 0$: each path-connected component of X meets A.

$n \geq 1$: (X, A) is 0-connected, and $\forall\ a_0 \in A$, $\forall\ 1 \leq k \leq n$, $\pi_k(X, A) = 0$

(4) A map $f : X \to Y$ in Top is called n-connected if $f_* : \pi_k(X, x) \to \pi_k(Y, f(x))$ is an iso for $0 \leq k \leq n-1$ and is surjective for $k = n$, for all $x \in X$.

(5) A map $f:(X,A)\longrightarrow(Y,B)$ in Top^2 is called n-connected (for $n\geqslant 1$) if for all $a\in A$, $\pi_k(X,A,a)\to\pi_k(Y,B,f(a))$ is an iso for $1\leq k\leq n-1$ and is surjective for $k=n$.

<u>**1.44 Theorem**</u> (Blakers-Massey or "Homotopy excision").

Let $X\in$ Top, U_0, U_1 open sub-spaces in X, with $X=U_0\cup U_1$. Let $U_{01}:=U_0\cap U_1$.

Suppose given $p,q\in\mathbb{N}$ such that

(a) $U_{01}\neq\emptyset$

(b) (U_0,U_{01}) is p-connected

(c) (U_1,U_{01}) is q-connected

Then the inclusion $(U_0,U_{01})\hookrightarrow(X,U_1)$ is $(p+q)$-connected.

For the proof, we first need a few lemma.

<u>**1.45 Lemma**</u> Let $(X,A)\in$ Top^2, $n\geqslant 1$. TFAE

(1) For all $a_0\in A$, $\pi_n(X,A,a_0)=0$

(2) Each map $f:(I^n,\partial I^n)\to(X,A)$ is homotopic, in Top^2 to a constant map.

(3) Each map $f:(I^n,\partial I^n)\to(X,A)$ is homotopic rel ∂I^n to a map into A.

<u>Def</u>: A homotopy $(X,A)\times I\xrightarrow{H}(Y,B)$ in Top^2 is a <u>homotopy relative to A</u> ($H_0\overset{H}{\simeq}H_1: X\to Y$ rel A) if $\forall a\in H$, $\forall t\in I$, $H(a,t)=H(a,0)$.

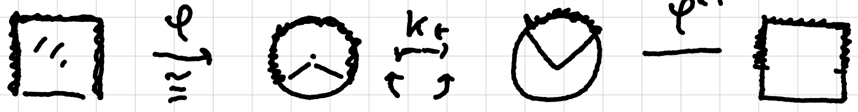
proof: We only give a sketch; with formulas as an exercise.

(1) $\Rightarrow$ (2) Let $f:(I,\partial I^n)\to(X,A)$ be given. There exists $id\overset{H}{\simeq}h:(I^n,\partial I^n)\longrightarrow(I^n,\partial I^n)$ with $h(J^{n-1})=\{(1,\dots,1,\frac{1}{2})\}$. Then $f\simeq f\circ h$ in Top^2, and

and $f \circ h : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, a_0)$ $(a_0 = f(1, \dots, 1, \frac{1}{2}))$

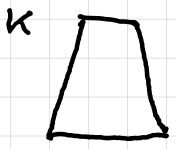
So $f \circ h \simeq c_{a_0}$ by (1).

H_t :



(2) ⇒ (3) Again, given $f : (I^n, \partial I^n) \to (X, A)$, assuming (2) we have a homotopy $H : I^n \times I \to X$ H_1 constant. It suffices to precompose H with the following map $g : I^n \times I \circlearrowleft$:

Let K be the convex hull of $I^n \times \{0\}$ and $[\frac{1}{4}, \frac{3}{4}]^n \times \{1\}$ in $I^n \times I$; Then g is the $u \circ \sigma$ where



$\sigma : I^n \times I \longrightarrow K$, $\sigma|_K = id_K$, and for $x \notin K$, $\sigma(x)$ is the vertical projection on ∂K.

and u expands back to $I^n \times I$ "horizontally".

(3) ⇒ (1) is obvious: use $\pi_n(A, A, a_0) = 0$! □

<u>Notation</u>: Say that a <u>cube</u> W in $\mathbb{R}^n$ is a subset $W = W(a, \delta, L)$ for some fixed $a \in \mathbb{R}^n$, $\delta > 0$, $L \subset \{1, \dots, n\}$:

$W = \{x \in \mathbb{R}^n ;\ a_i \leq x_i \leq a_i + \delta \text{ if } i \in L,\ a_i = x_i \text{ if } i \notin L\}$.

Here $\dim(W) = \text{card}\, L$. For $0 \leq k < \dim W$, a proper face of W is a "sub cube" W' of W determined by subsets $L_0, L_1 \subset L$, $L_0 \cap L_1 = \emptyset$, $k = \text{card}(L \setminus (L_0 \cup L_1))$.

$W' = \{x \in W ;\ x_i = a \text{ if } i \in L_0,\ x_i = a_i + \delta \text{ if } i \in L_1\}$

Note that $W' = W(a', \delta, L')$ for some $a \in \mathbb{R}^n$, $L' \subset L$.

$\partial W :=$ union of all proper subfaces.

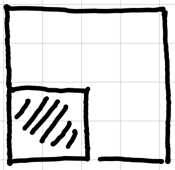
For $W = W(a, \delta, L)$ and $1 \leq p \leq \dim W$, let

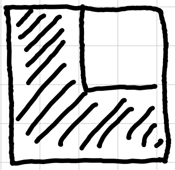
$K_p(W) = \{x \in W ;\ x_i \leq a_i + \delta/2 \text{ for at least } p \text{ values } i \in L\}$

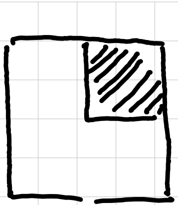
$G_p(W) = \{$ —— " —— $\geq$, ———— " ———— $\}$

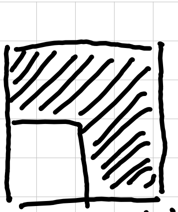
and $K_p(W) = \emptyset = G_p(W)$ if $p > \dim(W)$.

Example: in I^2,


$K_2(I^n)$


$K_1(I^n)$


$G_2(I^n)$

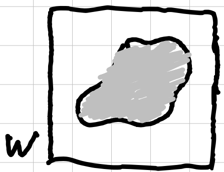

$G_1(I^n)$

1.46 lemma: Suppose given $f: W \to X$ and $A \subset X$, and some $p \leq n$ such that for any proper face W' of W,

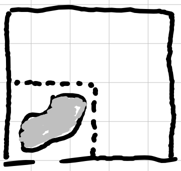
$$f^{-1}(A) \cap W' \subset K_p(W').$$

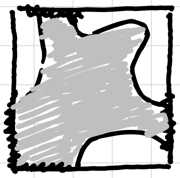
Then $\exists\, g: W \to X$, $g \simeq f$ rel ∂W and $g^{-1}(A) \subset K_p(W)$ (+ same statement replacing K_p by G_p).

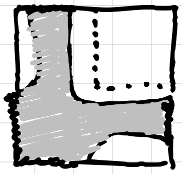
Eg If $n = 2$, $p = 2$: | If $n = 2$, $P = 1$



$f^{-1}(A)$

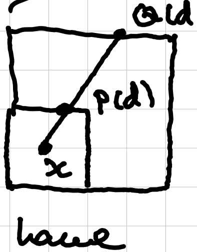

$g^{-1}(A)$


$f^{-1}(A)$


$g^{-1}(A)$

Proof: Assume $W = I^n$ (use affine iso). Id_{I^n} is homotopic to $h: I^n \to I^n$ rel (∂I^n), where h is constructed as follows.

Take $x = (\frac{1}{4}, \dots, \frac{1}{4})$. Take $h(x) = x$, and for any half line d out of x, let



we have
<u>$h \simeq id$ rel ∂I^n</u>

$\left.\begin{matrix} P(d) \\ Q(d) \end{matrix}\right\}$ be the intersection of d with $\left\{\begin{matrix} \partial [0, 1/2]^n \\ \partial I^n. \end{matrix}\right.$

Then h maps affinely the segment $\overrightarrow{x P(d)}$ onto $\overrightarrow{x Q(d)}$ and the segment $\overrightarrow{Q(d) P(d)}$ onto $Q(d)$. Note that $h(z) = x + t(z - x)$ for some value of t depending on x. Let $g = f \circ h$. Take $z \in I^n$ such $g(z) \in A$. (i) If $z_i < \frac{1}{2}$ $\forall i$, then $z \in K_p(I^n)$.
(ii) $\exists\, i$ $z_i \geqslant 1/2$. Then $h(z) \in f^{-1}(A) \cap W'$ for some

proper face W' of I^n. Thus $h(z) \in K_p(W')_p$ so for (at least) p values of i, $h(z)_i < \frac{1}{2}$. since z moves "away" of x.
Since $h(z)_i = \frac{1}{4} + t(z_i - \frac{1}{4})$ for some $t \geqslant 1$, $z_i < \frac{1+t}{t} 1/4$, and $\frac{1+t}{t} \leqslant 2$ for $t \geqslant 1$. Thus $z \in K_p(I^n)$. This proves $g^{-1}(A) \subset K_p(I^n)$.
The proof for G_p follows (e.g. use a reflection!). $\boxtimes$

Suppose now X, U_0, U_1 as in 1.44. Suppose $f : I^n \to X$ given and subdivide I^n in smaller cubes K such that for each such K, $f(K) \subset U_0$ or $f(K) \subset U_1$. Then:

1.47 Lemma: $\exists\, f \overset{H}{\simeq} g : I^n \to X$, such that for any W in the collection of subcubes K as above, or their proper faces:

(1) If $f(W) \subset U_{01}$, then $H_t|_W = f|_W \ \forall\, t \in [0,1]$.
(2) If $f(W) \subset U_i$, then $H_t(W) \subset U_i \ \forall\, t \in [0,1]$, $i = 0, 1$

(3) If $f(W) \subset U_0$, then $g^{-1}(U_0 \setminus U_{01}) \cap W \subset K_{p+1}(W)$
(4) If $f(W) \subset U_1$, then $g^{-1}(U_1 \setminus U_{01}) \cap W \subset G_{q+1}(W)$.

proof: We construct by induction on $0 \leq k \leq n$ a sequence of homotopies $f = f_{-1} \overset{H^0}{\simeq} f_0 \overset{H^1}{\simeq} f_1 \simeq \dots \overset{H^n}{\simeq} f_n : I^n \to X$ such that: (a) (1)+(2) hold for $(f, H) = (f_{k-1}, H^k)$.
(b) (3) and (4) hold for f_k whenever $\dim(W) \leq k$.
Then, taking the concatenation of homotopies $H = H^0 * \dots * H^n$ and $g = f_n$, we have proven the lemma.

$k=0$: let C^0 be the collection of subcubes of dim 0 in I^n.
We first define H^0 on $\bigcup_{W \in C^0} W \times I \hookrightarrow I^n \times I$:
If $f(W) \subset U_{01}$, take $H^0_t|_W = f|_W$.
If $f(W) \subset U_0$ but $f(W) \not\subset U_1$, take $H^0_t|_W$ a path from W to U_{01} in U_0 ($\exists$ since (U_0, U_{01}) 0-connected)

Thus $f_0(W) \subset U_{01}$ and $f_0^{-1}(U_0 \setminus U_{01}) \cap W = \emptyset$ so (3) holds. Similarly, if $f(W) \subset U_1$ but $f(W) \not\subset U_0$, take $H^0|_W$ a path from $f(W)$ to U_{01} in U_1. Final step (★): extend $H^0|_{(\bigcup_{W \in C_0} W) \times I}$ to $I^n \times I$ inductively on the dimension of subcubes W:

Take $(H^0|_{W \times I})_t = f|_W$ if $f(W) \subset U_{01}$, and otherwise use the retraction $W \times I \xrightarrow{r} (W \times \{0\}) \cup (\partial W \times I)$ (H^0 already defined here by induction).

Suppose $1 \leq k \leq n$ and $H^0, \dots, H^{n-}$ constructed.

Take $W \in C^k$, the collection of faces of subcubes of dim k.

If $f_{k-1}(W) \subset U_{01}$, take $H^k_t|_W = f_{k-1}|_W$

(1) Suppose $f_{k-1}(W) \subset U_0$ and $f_{k-1}(W) \not\subset U_1$.

(a) Suppose $k \leq p$. In particular $K_{p+1}(W) = \emptyset$, so we will need $f_k(W) \subset U_{01}$. Note that $K_{p+1}(W') = \emptyset$ for any proper face W' of W, so $f_{k-1}(\partial W) \subset U_{01}$.

Since (U_0, U_{01}) is p-connected, by 1.45 we can find $H^k|_W : W \times I \to U_0$ rel (∂W) with $H^k_1|_W(W) \subset U_{01}$

(b) Suppose $k \geq p+1$. By induction hypothesis, for any face W' of W, we have (3), thus

$$f_{k-1}^{-1}(U_0 \setminus U_{01}) \cap W' \subset K_{p+1}(W')$$

Applying lemma 1.46, we can find a homotopy $H^k|_W : W \to U_0$ rel (∂W) with $(H^k_1|_W)^{-1}(U_0 \setminus U_{01}) \cap W \subset K_{p+1}(W)$. Finally extend to $I^n \times I$ with (★) as above.

(2) If $f_{k-1}(W) \subset U_1$, $f_{k-1}(W) \not\subset U_0$, we apply the same two steps (a), (b), replacing K_{p+1} with G_{q+1}, to define $H^k|_W$. When $H^k|_W$ has been defined for all $W \in C^k$, extend to $I^n \times I$ using (★). This ends the induction #

Note: H^k is relative W' for any cube W' of dim $< k$.

1.48 Lemma: The inclusion $F(U_0, U_0, U_{01}) \hookrightarrow F(U_0, X, U_1)$ is $(p+q-1)$-connected. $\quad \overset{\shortparallel}{F_{U_0}} \quad \overset{\shortparallel}{F_X}$

proof: Chose $n \leq p+q-1$ and $(I^n, \partial I^n) \xrightarrow{\alpha} (F_X, F_{U_0})$. It suffices to find a htpy $\alpha \overset{H}{\simeq} \beta$ of pairs with $\beta(I^n) \subset F_{U_0}$ (see Exercise 3.8). Now $\alpha: I^n \to F_X$ adjoint to $\check{\alpha}: I^n \times I \to X$. By def, $\check{\alpha}$ satisfies the following conditions (say $\check{\alpha}$ admissible)

$$\left.\begin{array}{l} \check{\alpha}(x,0) \in U_0 \\ \check{\alpha}(x,1) \in U_1 \end{array}\right\} \forall x \in I^n, \text{ since } \alpha(I^n) \subset F_X$$

$$\check{\alpha}(x,t) \in U_0 \Big\} \forall (x,t) \in \partial I^n \times I, \text{ since } \alpha(\partial I^n) \subset F_{U_0}$$

It suffices to show that $\check{\alpha}$ is homotopic, within admissible maps, to $\check{\beta}: I^n \times I \to X$ with $\check{\beta}(I^n \times I) \subset U_0$.

Apply 1.47 to $\check{\alpha}$, obtaining $\gamma: I^n \times I \to X$.

Note that the homotopies of 1.47 preserve "admissibility'

because of requirement 1.47. (2).

Let $\pi: I^n \times I \to I$ be the projection onto first factor.

Let $A_i = \pi(\gamma^{-1}(X \setminus U_i))$ for for $i = 0, 1$.

Then $A_0 \cap A_1 = \emptyset$. Indeed, suppose $z \in \gamma^{-1}(X \setminus U_0) \cap W$ for some $(n+1)$ dim'l cube of $I^n \times I$ used in 1.47.

Then $z \in K_{p+1}(W)$ has at least $(p+1)$ "small" coordinates, and $\pi(z)$ at least p. Similarly, $x \in A_1$ has at least q "large" coordinates. Since $n < p+q$, we have $A_0 \cap A_1 = \emptyset$.

Not also $\partial I^n \cap A_0 = \emptyset$; By Urysohn, can chose $u: I^n \to I$ with $u(A_0) = \{0\}$ and $u(\partial I^n \cup A_1) = \{1\}$.

Then
$$H: (I^n \times I) \times I \longrightarrow X$$
$$(x, t, s) \longmapsto \gamma(x, (1-s)t + s\,t\,u(x))$$
is a homotopy within admissibles maps, and take $\check{\beta} = H_1$. ▨

proof of Blakers-Massey. We have a diagram of fiber sequences

$$\begin{tikzcd}
\overset{=F'_{U_0}}{F(\{x_0\},U_0,U_{01})} \arrow[r] \arrow[d, hook, "j"] & \overset{=F_{U_0}}{F(U_0,U_0,U_{01})} \arrow[r, "\varepsilon_0"] \arrow[d, hook, "i"] & U_0 \arrow[d, "="] \\
\overset{=F'_X}{F(\{x_0\},X,U_1)} \arrow[r] & \overset{=F_X}{F(U_0,X,U_1)} \arrow[r] & U_0
\end{tikzcd}$$

for any choice of $x_0 \in U_0$. Taking long htpy sequences (and omitting x_0 from the notation), we get:

$$\begin{tikzcd}
\pi_{k+1}(F_{U_0}) \arrow[r] \arrow[d, "i_*"] & \pi_{k+1}(U_0) \arrow[r, "\partial"] \arrow[d, "="'] & \pi_k(F'_{U_0}) \arrow[r] \arrow[d, "j_*"] & \pi_k(F_{U_0}) \arrow[r] \arrow[d, "i_*"] & \pi_k(U_0) \arrow[d, "="'] \\
\pi_{k+1}(F_X) \arrow[r] & \pi_{k+1}(U_0) \arrow[r, "\partial"] & \pi_k(F'_X) \arrow[r] & \pi_k(F_X) \arrow[r] & \pi_k(U_0)
\end{tikzcd}$$

By 1.48, i_* is an iso if $k \leq p+q-2$, surj. if $k = p+q-1$. By the 5 lemma (refined version), the same holds for j_*. Conclude by using

$$\begin{tikzcd}
\pi_k(F(\{x_0\},U_0,U_{01}),*) \arrow[r, "\cong"] \arrow[d] & \pi_{n+1}(U_0,U_{01},x_0) \arrow[d] \\
\pi_k(F(\{x_0\},X,U_1),*) \arrow[r, "\cong"] & \pi_{n+1}(X,U_1,x_0)
\end{tikzcd}$$

(analogous to 1.41). ▨

1.49 Def : For $(X, x_0) \in Top_*$, define the suspension

$$(\Sigma X, *) = \left(X \times I \big/ (X \times \partial I) \cup \{x_0\} \times I , [x_0] \right) \in Top_*.$$

For any $n \geq 0$, the suspension map

$$\Sigma_* : \pi_n(X, x_0) \to \pi_{n+1}(\Sigma X, *)$$ is defined

as follow : for $f : (I^n, \partial I^n) \to (X, x_0)$, let $\Sigma_*([f])$ be the class of the map $\tilde{f} : (I^{n+1}, \partial I^{n+1}) \to (\Sigma X, *)$ obtained by taking the quotient of

$f \times I : (I^n \times I, \partial I^n \times I) \to (X \times I, \{x_0\} \times I)$:

$$\begin{tikzcd}
(I^n \times I, \partial I^n \times I) \arrow[r, "f \times I"] \arrow[d] & (X \times I, \{x_0\} \times I) \arrow[d, "q"] \\
(I^{n+1}, \partial(I^{n+1})) \arrow[r, dashed, "\tilde{f}"] & (\Sigma X, *)
\end{tikzcd}$$

1.50 Exercise: (a) Check Σ_* is a group homom. if $n \geq 1$

(b) Show that we have a htpy equivalence $\Sigma S^n \to S^{n+1}$.

1.51 Corollary (of BM)

(a) For $n \geq 1$, S^n is $(n-1)$-connected.

(b) $\Sigma_* : \pi_k(S^n, *) \to \pi_{k+1}(S^{n+1}, *)$ is
- an isomorphism if $k \leq 2n-2$
- surjective if $k = 2n-1$

(c) We have $\Sigma_* : \pi_1(S^1, *) = \mathbb{Z} \twoheadrightarrow \pi_2(S^2, *)$, and $\Sigma_* : \pi_n(S^n, *) \to \pi_{n+1}(S^{n+1}, *)$ an iso if $n \geq 2$.

Proof: Take $U_0 = S^{n+1} \setminus \{N\}$, $U_1 = S^{n+1} \setminus \{S\}$, $N\ (S)$ is the north (South) pole. Then $U_{01} = U_0 \cap U_1 \cong S^n \times \mathring{I} \simeq S^n$

(a) Induction on $n \geq 1$
- $\underline{n=1}$ S^1 is path connected ($=$ 0-connected), ok!
- $\underline{\text{Assume } n \geq 1}$ and proven that S^n is $(n-1)$-connected.

By the htpy seq of the pair $(U_0, U_{01}, *)$, using $U_0 \simeq *$, we have

$$\begin{tikzcd}
0 \arrow[r] & \pi_{k+1}(U_0, U_{01}, *) \arrow[r, "\partial_k", "\cong"'] & \pi_k(U_{01}, *) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & 0 \\
& & \pi_k(S^n, *) &
\end{tikzcd}$$

for all $k \geq 1$.

Thus (U_0, U_{01}) is n-connected.

By symmetry, (U_1, U_{01}) is also $(n-1)$-connected.

By BM, we deduce that $(U_0, S^n) \hookrightarrow (S^{n+1}, U_1)$ is $2n$-con.

thus $\pi_k(S^n, *) \xleftarrow[\partial_k]{\cong} \pi_{k+1}(U_0, S^n, *) \xrightarrow{j_*} \pi_{k+1}(S^{n+1}, U_1, *)$ $\xleftarrow[i_*]{\cong} \pi_{k+1}(S^{n+1}, *)$ and j_* is an iso if $k+1 \leq 2n-1$ and surjective if $k+1 = 2n$.

In particular, this proves that if $k \leq 2n-1$, $\pi_k(S^n,*) \to \pi_{k+1}(S^{n+1},*)$ is surjective.

$\pi_{k+1}(S^{n+1},*) = 0$ for $k+1 \leq n$, and S^{n+1} is n connected!

(b) It suffices to note that

$$\Sigma_* = i_*^{-1} \circ j_* \circ \partial_{k+1}^{-1} \text{ above.}$$

(c) Just a special case of (b)!

1.52 Theorem For $n \geq 1$, we have an isomorphism

$$\mathbb{Z} \to \pi_n(S^n,*)$$
$$1 \mapsto [id_{S^n}].$$

proof: Known for S^1 (Eg Fibration $\mathbb{Z} \to \mathbb{R} \to S^1$).
Since S^3 is 2 connected, the Hopf fibration $S^1 \to S^3 \to S^2$ gives an exact sequence $0 \to \pi_2(S^2,*) \to \pi_1(S^1,*) \to 0$.

We deduce that $\pi_2(S^2,*) = \mathbb{Z}$, and since $\pi_1(S^1,*) \xrightarrow{\Sigma_*} \pi_2(S^2,*)$ is surjective, it is an iso.
By 1.51 (c) we conclude. #

1.53 Theorem: $\pi_3(S^2;*) \cong \mathbb{Z}$, generated by the Hopf map. h

proof: We know that $\pi_k(S^1,*) = 0$ if $k \geq 2$.
Thus $\pi_3(S^3,*) \xrightarrow{h_*} \pi_3(S^2,*)$ is an iso. ▨
$[id] \mapsto [h]$

1.54 Remark We see from 1.51 (b) that if $k \leq n-2$

$$\pi_{k+n}(S^n,*) \xrightarrow{\Sigma} \pi_{k+n+1}(S^{n+1},*) \xrightarrow{\Sigma} \dots$$

are all isos, so does not depend on $n \geq k+2$.

$\pi_k^s := \pi_{k+n}(S^n,*)$ for $n \geq k+2$ is called the s'th stable homotopy group of spheres.

Theorem 1.52 tells us that $\pi_0^s \cong \mathbb{Z}$, and Theorem 1.52 tells us that we have a surjection $\mathbb{Z} \to \pi_1^s$ since $\mathbb{Z} = \pi_2 S^3 \twoheadrightarrow \pi_3 S^4 \cong \pi_4 S^5 \cong \dots$

k	0	1	2	3	4	5	6	7	8	9	10	11	12
π_k^s	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	2	240	4	8	6	504	0

A few words on the chromatic picture and the telescope conjecture...